Examination Performance Analysis (et4-367)

Question 1 : Derive the famous Pollaczek-Khinchin formula for the the probability generating function of the system content in an M/G/1 queue, $S(z) = (1-\rho)\frac{(z-1)\varphi_x(\lambda-\lambda z)}{z-\varphi_x(\lambda-\lambda z)}$

Question 2 : A faulty digital video conferencing system shows a clustered error pattern. If a bit is received correctly, then the chance to receive the next bit correctly is 0.999. If a bit is received incorrectly, then the next bit is incorrect with probability 0.95.

1. Model the error pattern of this system using the discrete-time Markov chain.
2. How many communication classes does the Markov chain have? Is it irreducible?
3. In the long run, what is the fraction of correctly received bits and the fraction of incorrectly received bits?
4. After the system is repaired, it works properly for 99.9% of the time. A test sequence after repair shows that, when always starting with a correctly received bit, the next 10 bits are correctly received with probability 0.9999. What is the probability now that a correctly (and analogously incorrectly) received bit is followed by another correct (incorrect) bit?

Question 3 : Long time ago telephone exchanges consisted of one to several hundred plug boards staffed by telephone operators. In one city, a telephone operator of that time had 18 calls between 8.00 and 11.00. On her plug board, she had several signalization lamps and for every call a lamp was turned on and after the call was transferred it went off.

If she goes to a coffee break at 10.00 and she drinks coffee for 10 minutes, calculate under the assumption that calls have a Poisson distribution of arrivals :

1. number of lamps that are on when the telephone operator comes back.
2. probability that not a single lamp is on.
3. probability that more than 10 lamps are on.
4. the most probable number of lamps that are on.
5. if she has found 1 lamp on, the most probable minute while she was away at which the lamp turned on.
6. if she has found all lamps off, the probability that one lamp will be on in a minute.

Question 4 : Multiple Choices

A. Suppose that we have the option to build a system S_s with all subsystems in series (in one line) and a system S_p with all subsystems in parallel. Suppose also the reliability function of each subsystem is the same and equal to $0 < R(t) < 1$. Which of the two designs S_s and S_p has the highest reliability function?

1. the series-system S_s
2. the parallel-system S_p
3. not possible to make this choice, because it depends on $R(t)$.

B. The Lindley integral equation for the waiting time in the GI/G/1 queue,

$$W(x) = \int_{-\infty}^{x} W(x-s)dC(s) \qquad \text{if } x \geq 0 \qquad (1)$$
$$= 0 \qquad \text{if } x < 0$$

where $W(x) = \lim_{n\to\infty} \Pr[w_n < x]$ with $C(x) = \lim_{n\to\infty} C_n(x)$ is valid

1. only for queueing systems that are nul-recurrent Markov chains
2. only if the steady-state exists
3. always
4. none of the above answers is correct

C. There exists a source-destination pair in a network with $N = 7$ nodes that contains 330 different paths

1. correct
2. wrong
3. not enough data to make any conclusion

D. The shortest path model, resulting in a uniform recursive tree, is applicable

1. in any graph, provided the number of links and nodes is large $N \to \infty$
2. only asymptotically for large N, in graphs with a regular link weight distribution and "sufficiently dense", i.e. the link density $p > p_c \sim \frac{\log N}{N}$
3. in the complete graph K_N with binomial link weights with $E[w] = 1$ and $\text{Var}[w]/E[w] \to 0$ for large N
4. none of the above answers is correct

1 Solutions

Question 1 : book p. 285-286 + 289-290.

Question 2: book p. 505-506.

Question 3: The probability that $k \in [0, \infty)$ users will arrive in the interval τ is Poissonean,

$$\Pr[X(t+\tau) - X(t) = k] = \frac{(\lambda\tau)^k}{k!} e^{-\lambda\tau} \tag{2}$$

with $E[X(t)] = \lambda t$. The main observations in this problem occur between 10.00 and 10.10. Hence, $t = 10h00$ and $\tau = 10\,\text{min}$. The stationarity of the Poisson process and $X(0) = 0$ tells that

$$\Pr[X(t+\tau) - X(t) = k] = \Pr[X(\tau) = k]$$

The rate of arrivals can be found from the fact that 18 calls arrive during 3 hours,

$$\lambda = \frac{18 \text{ arrivals}}{3 \cdot 60\,\text{min}} = \frac{1}{10} \text{ arrivals}/\,\text{min}$$

and $\lambda\tau = 1$.

1. Using the formula (2), the possible number of arrivals in the interval τ can range from 0 to ∞, so the operator can find any integer number of on lamps from 0 on.

2. Using the formula (2), the probability for $k = 0$ is

   $$\Pr[X(\tau) = 0] = e^{-1}$$

   The statement "not a single lamp" is somewhat misleading. It can be interpreted as

   $$\Pr[X(\tau) \neq 1] = 1 - \Pr[X(\tau) = 1] = 1 - e^{-1}$$

   Both answers have been regarded as correct.

3. The probability that more than 10 lamps are on is

   $$\Pr[X(\tau) > 10] = 1 - \Pr[X(\tau) \leq 10] = 1 - e^{-1} \sum_{k=0}^{10} \frac{1}{k!} \simeq 10^{-8}$$

4. For $\lambda\tau = 1$ and the fact that $1/k!$ is maximal for $k = 0$ or 1, (2) shows that the maximal probability is reached when $k = 0$ or 1.

5. The Poisson process is memoryless and the size of the observation interval is important, not the position of that interval in the whole observation time. We can conclude that all the minutes, while an operator was absent, are equaly probable.

6. Because of the memoryless property we are interested only in the size of the interval which is 1 minute. Based on the formula (2), we have with $t_1 = t + \tau = 10h10$ and $\tau_1 = 1$ min,

$$\Pr\left[X(t_1+\tau_1) - X(t_1) = 1 | X(t+\tau) - X(t) = 0\right] = \Pr\left[X(t_1+\tau_1) - X(t_1) = 1\right]$$

by the independence of non-overlapping intervals. By the stationarity of the Poisson process, we have

$$\begin{aligned}\Pr\left[X(t_1+\tau_1) - X(t_1) = 1\right] &= \Pr\left[X(\tau_1) = 1\right] \\ &= \lambda\tau_1 e^{-\lambda\tau_1} = \frac{1}{10}e^{-\frac{1}{10}} \simeq 0.0905\end{aligned}$$

Question 4 : Multiple Choices

A. Answer is 2: the parallel-system S_p. From the solutions on p. 500, we need to compare $R_s = R^n$ and $R_p = 1 - (1-R)^n$. Since $0 < R < 1$, the extreme values where $R_s = R_p$ are excluded. By drawing the functions or investigating the sign $f(R) = R_p - R_s$, we find that $f(R) > 0$ for $0 < R < 1$.

B. Answer is 4: Lindley equation is only valid for queueing systems with independent arrival and service processes and when the traffic intensity $\rho < 1$.

C. Answer is 2: the maximum number of different path between any source-destination is $[e(N-2)!]$ (see p. 322). For $N = 7$, $[e5!] = 326 < 330$.

D. Answer is 4, but 2 (that forgot to mention that link weights need to be i.i.d.) was also awarded: book p. 348 en p. 354.