Examination Performance Analysis (et4-032)

Question 1 (Queuing): A certain company hires a service of a call center with two phone lines. It has been determined based on measurements that both lines are busy 20% of the time and that the average call holding time is 10 minutes. In order to improve this situation the company is considering two different alternatives: (1) either choose for a different service package offered by the call center with an average holding time of 8 minutes or (2) by using a third additional phone line without changing the current service of the call center.

(a) Which one of the alternatives leads to the best outcome? Calculate the call blocking probability for both the cases.

(b) Calculate the intensity of the carried traffic in both cases.

(Tip: Model the system as a pure loss system receiving Poisson call arrivals with a constant rate).

Question 2 (Markov chain):

The causes for a server failure can be classified as hidden and evident. The times of occurrence of hidden and evident failures is exponentially distributed, with mean values of 400 h and 100 h respectively. When the evident failure takes place, the server is repaired. The mean time needed to repair a server is 10 h. In case of a hidden fault, the server remains out of service until an evident attack occurs, after which it is repaired.

(a) Draw a state-space diagram and determine the infnitesimal generator matrix Q.

(b) Determine the steady state probabilities.

(c) What is the asymptotic unavailibility of the server? (Asymptotic unavailibility is the probability that the server is down when $t \to \infty$)

(d) What is the average number of system failures in a total time of one year (of 365 days)?

Question 3 (Poisson):

A link of a packet network carries on the average 10 packets per second. The packets arrive according to a Poisson process. A packet has a probability of 30 % to be an acknowledgment (ACK) packet independent of the others. The link is monitored during an interval of 1 second.

(a) What is the probability that at least one ACK packet has been observed?

(b) What is the expected number of all packets given that 5 ACK packets have been spotted on the link?

(c) Given that 8 packets have been observed in total, what is the probability that two of them are ACK packets?

Question 4 (Poisson): Proof the following theorem: Given that exactly one event of a Poisson process $\{X(t); t \geq 0\}$ has occurred during the interval [0,t], the time of occurrence of this event is uniformly distributed over [0,t]

Question 5 (Queuing): Derive the famous Pollaczek-Khinchin formula for the the probability generating function of the system content in an M/G/1 queue:

$$S(z) = (1-\rho)\,\frac{(z-1)\,\varphi_x(\lambda - \lambda z)}{z - \varphi_x(\lambda - \lambda z)}$$

1 Solutions

Question 1 (queuing):

A certain company has a call center with two phone lines. It has been determined based on measurements that both lines are busy 20% of the time. It has also been determined that the average call holding time was 20 minutes. In order to improve this situation the company is considering two different alternatives. If they would change services offered by the center, the average holding time would reduce to 15 minutes. On the other hand, by opening a third phone line they could maybe keep the services unchanged.

(a) Which one of the alternatives leads to better outcome? Calculate the call blocking probability for both the cases.

(b) Calculate the intensity of the carried traffic for both the cases.

(Tip: Model the system as a pure loss system receiving Poisson call arrivals with a constant rate. Traffic intensity is defined as $\rho = \frac{\lambda}{\mu}$).

Solution 1:

(a) The average call holding time is $\frac{1}{\mu}$ = 20 minutes, the time blocking probability P_{Bt} = 0.2. Since arrivals are Poissonian, due to the PASTA property the time blocking probability P_{Bt} is equal to P_{Bc}, where with P_{Bc} we denote the call blocking probability. Since the number of lines is $n = 2$, the traffic intensity $\rho = \frac{\lambda}{\mu}$ can be calculated using *Erlang's B formula* (13.17):

$$P_B = P_R\left[N_S = m\right] = \frac{\frac{\rho^m}{m!}}{\sum_{j=0}^{m} \frac{\rho j}{j!}}$$

for $n = 2$ we get

$$P_B = P_R\left[N_S = 2\right] = \frac{\frac{\rho^2}{2!}}{1 + \rho + \rho^2/2}$$

Solving this quadratic equation results in $\rho = 1$, $\lambda = 0.05$

If the average call duration decreased to 15 minutes, $\rho = 0.75$, and the call blocking probability would be

$$P_{B1} = P_R\left[N_S = 2\right] = \frac{\frac{\rho^2}{2!}}{1 + \rho + \rho^2/2!} = 0.13$$

On the other hand, including an additional line would lead to

$$P_{B2} = P_R\left[N_S = 3\right] = \frac{\frac{\rho^3}{3!}}{1 + \rho + \rho^2/2! + \rho^3/3!} = 0.06$$

Since $P_{B2} < P_{B1}$ creating a third phone line would lead to an improved system.

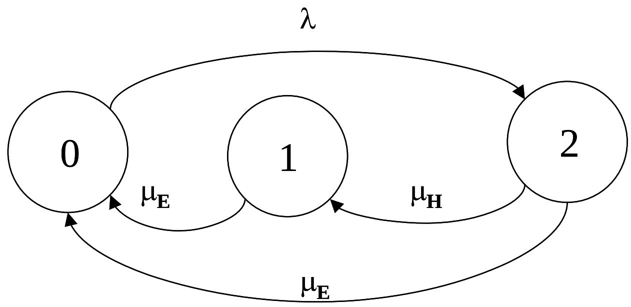

Figure 1:

(b) The carried traffic is the traffic that is not blocked, so

$$\rho_c = \rho(1 - P_B)$$

which results in $\rho_{c1} = 0.65$ and $\rho_{c2} = 0.94$

Question 2 (Markov chain)

The causes for a server failure can be classified as hidden and evident. The times of occurrence of hidden and evident failures is exponentially distributed, with mean values of 400 h and 100 h respectively. When the evident failure takes place, the server is repaired. The mean time needed to repair a server is 10 h. In case of a hidden fault, the server remains out of service until an evident attack occurs, after which it is repaired.

(a) Draw a state-space diagram and determine the infnitesimal generator matrix Q.

(b) Determine the steady state probabilities.

(c) What is the asymptotic unavailibility $\mathbf{U}$ of the server? (Asymptotic unavailibility is the probability that the server is out of order when $t \to \infty$)

(d) What is the average number of system failures in a total time of one year?

Solution 2:

(a) The system states can be denoted in the following way:

0- the server fails due to an evident failure

1- the server fails due to a hidden failure

2- the server runs (no failures)

$\lambda = \frac{1}{10}h^{-1}, \mu_E = \frac{1}{100}h^{-1}, \mu_H = \frac{1}{400}h^{-1}$

The corresponding matrix $\mathbf{Q}$ is:

$$\mathbf{Q} = \begin{bmatrix} -\lambda & 0 & \lambda \\ \mu_E & -\mu_E & 0 \\ \mu_E & \mu_H & -\mu_H - \mu_E \end{bmatrix}$$

(b) The steady state vector $\boldsymbol{\pi}$ is solution of $\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}$ (10.19).

So,

$$\begin{bmatrix} \pi_0 & \pi_1 & \pi_2 \end{bmatrix} \begin{bmatrix} -\lambda & 0 & \lambda \\ \mu_E & -\mu_E & 0 \\ \mu_E & \mu_H & -\mu_E - \mu_H \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

In order to solve this set of equations, since the system is undetermined, we add the normalization condition $\sum_{i=0}^{2} \Pi_i = 1$,so we obtain:

$$\begin{bmatrix} \pi_0 & \pi_1 & \pi_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & \lambda \\ 1 & -\mu_E & 0 \\ 1 & \mu_H & -\mu_E - \mu_H \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

The steady state probabilities are then:

$$\pi_0 = \frac{\mu_E}{(\lambda + \mu_E)} = 0.091$$

$$\pi_1 = \frac{\lambda \mu_H}{(\lambda + \mu_E)(\mu_E + \mu_H)} = 0.182$$

$$\pi_2 = \frac{\lambda \mu_E}{(\lambda + \mu_E)(\mu_E + \mu_H)} = 0.727$$

(c) The asymptotic unavailability **U** is probability that the server is out of order when $t \to \infty$:

$$\mathbf{U} = \pi_0 + \pi_1 = 0.273$$

(d) The total rate of system failures r is the departure rate from state 2 to state 1, plus the departure rate from state 2 to state 0.

$$r = \pi_2 (\mu_E + \mu_H) = 0.009$$

If we denote with X the random variable of total failures over the period of 1 year, then the average value of X will be

$$E[X] = r \times 24 \times 365 = 79.6$$

Question 3 (Poisson): this question is exactly copied from book (p. 327 ex. 9): solution in on 338 ex. 9.