Exam Dynamische Regelsystemen 1 (SC2030et-D1) and Modeling and Control (SC4180es)

Wednesday 17 June 2009, 14:00 until 17:00

This exam consists of 5 open problems + a bonus problem. A formula sheet is attached.

Read every question very carefully before answering. Make sure that you motivate your answers and write clearly!

The maximum amount of points you can obtain for the regular exam is 100. The bonus problem at the end of the exam will allow for 5 extra points. The final grade is determined by (total points)/10 with a maximum of 10.

It is NOT allowed to use books, (handwritten) notes, or scientific calculators!

Good luck!

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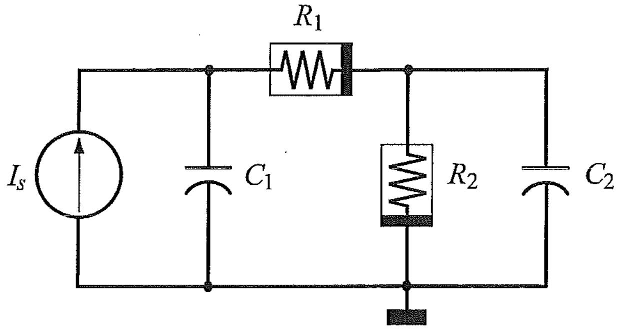
Problem 1 (20 points)

Consider the following nonlinear electrical RC network driven by a current source I_s:



Assume that the nonlinear resistors have a constitutive relationship of the form $i_{R_k} = \sqrt{u_{R_k}}$, for $k = 1, 2$. The capacitors C_1 and C_2 are linear and time-invariant (LTI).

a) Give an expression for the total co-energy in the network.

b) Give an expression for the total co-content.

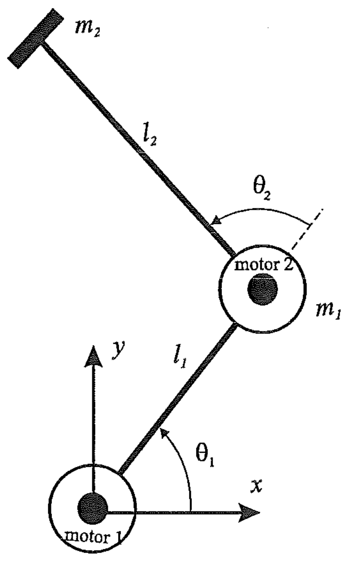
c) Draw the bond graph for the network and define the relevant efforts and flows at each bond.

d) Take the capacitor voltages u_{C_1} and u_{C_2} as the state variables. Derive the state space equations for the network.

e) Give the mechanical analog of the network and label the mechanical elements by their corresponding electrical analogs I_s, C_1, C_2, etc..

Problem 2 (20 points)

Consider the schematic diagram of a two-link manipulator (a Scara robot, for example) in the horizontal x-y plane.



For $m_1 = m_2 = 8$ kg, $l_1 = 0.5$ m, and $l_2 = 1$ m, the kinetic co-energy of the manipulator has the form

$$T^*(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^T \begin{pmatrix} 12+8\cos\theta_2 & 8+4\cos\theta_2 \\ 8+4\cos\theta_2 & 8 \end{pmatrix} \dot{\theta},$$

where $\theta = (\theta_1\ \theta_2)^T$ represent the link angles, and $\dot{\theta} = (\dot{\theta}_1\ \dot{\theta}_2)^T$ the corresponding angular velocities. The system has two control inputs, $\tau = (\tau_1\ \tau_2)^T$, i.e., the torques in the two joints.

a) Show that the dynamics of the manipulator can be described by the Lagrangian equations

$$\begin{pmatrix} 12+8\cos\theta_2 & 8+4\cos\theta_2 \\ 8+4\cos\theta_2 & 8 \end{pmatrix} \ddot{\theta} + \begin{pmatrix} -4\dot{\theta}_2(2\dot{\theta}_1+\dot{\theta}_2)\sin\theta_2 \\ 4\dot{\theta}_1^2\sin\theta_2 \end{pmatrix} = \tau. \qquad (*)$$

b) Suppose that there is also friction in the joints (motor 1 and 2) that can be modeled by constant friction coefficients B_1 and B_2. Give the Rayleigh dissipation (mechanical content) function.

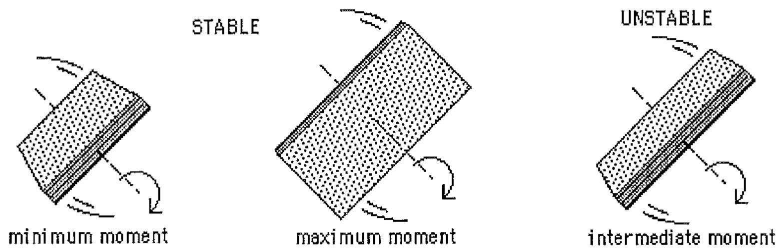
c) Show how the friction forces appear in the Lagrangian equations $(*)$.

d) Derive the expression for the Hamiltonian function.

e) Give an expression for the total power supplied by the motors.

Problem 3 (20 points)

When you get home after this exam, take one of your study books from the shelf and spin it about its three principle axis. For most books you will notice that the book will be stable when spinning about the axis with maximum and minimum moments of inertia, whereas it will be unstable when spinning about the axis with intermediate moment (then it will twist as well as spin).



In mathematical terms, the motion of the book can be approximated by the rigid body equations, which, in so-called port-Hamiltonian form, are given by

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \end{bmatrix} = \begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \\ \frac{\partial H}{\partial p_3} \end{bmatrix},$$

with Hamiltonian

$$H(p) = \frac{1}{2}\left(\frac{p_1^2}{I_1} + \frac{p_2^2}{I_2} + \frac{p_3^2}{I_3}\right),$$

momenta $p = (p_1, p_2, p_3)^T$, and moments of inertia $I_1 > I_2 > I_3 > 0$.

a) Determine the equilibrium point(s) $p^* = (p_1^*, p_2^*, p_3^*)^T$ of the system?

b) Show that the motion about the second axis (intermediate moment) is indeed unstable using Lyapunov's linearization (first) method. For convenience take $p^* = (0,1,0)^T$.

c) Show that $p^* = (0,0,0)^T$ is stable using Lyapunov's second (direct) method. [Hint: use the Hamiltonian as a (candidate) Lyapunov function.]

d) Is $p^* = (0,0,0)^T$ also asymptotically stable? Motivate your answer!

Problem 4 (20 points)

Consider the following system with input $u(t)$ and output $y(t)$ and given as

$$\frac{d^3y(t)}{dt^3} + 5\frac{d^2y(t)}{dt^2} + \alpha\frac{dy(t)}{dt} = \frac{du(t)}{dt} + 6u(t)$$

a) Determine the transfer function $H(s)$ of the system. For which values of α is the system stable? Is it also asymptotically stable?

b) Take $\alpha = 6$. Determine the impulse response $h(t)$ of the system.

c) Take again $\alpha = 6$ and use the convolution integral to determine the step response of the system? How are the impulse response $h(t)$ and step response $r(t)$ related?

d) Give an explicit expression for the magnitude and phase of the following second-order system (homogeneous response $sH(s)$)

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6 = 0.$$

Problem 5 (20 points)

Consider the state-space system

$$\frac{dx(t)}{dt} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & -2 & 0 \end{bmatrix} x(t).$$

a) Determine the transfer function $H(s)$. Is the system stable? Motive your answer!

b) When is a realization of a system minimal? Is the state-space realization (A,B,C) a minimal realization? Motive your answers!

c) Determine a minimal state space representation of this system?

d) Determine the undamped natural frequency ω_n and damping ratio ζ of the system $H(s)$. Indicate if the system is 'overdamped', 'critically damped' or 'underdamped'?

Bonus Problem (5 points extra)

Give the fluid mechanical analog system of the electrical circuit of Problem 1. [Hint: recall that the fluid analogy of voltage is pressure, whereas a capacitor corresponds to a tank.]

Formula sheet

Euler-Lagrange equations for mechanical systems

The Euler-Lagrange equations for a system without dissipation are given by

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}}(q,\dot{q})\right) - \frac{\partial \mathscr{L}}{\partial q}(q,\dot{q}) = F,$$

with Lagrangian $\mathscr{L}(q,\dot{q}) = T^*(q,\dot{q}) - V(q)$ and $F = F^e$. Here T^* is the kinetic co-energy, i.e., $T^*(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q}$, and V is the potential energy. In vector notation with m inputs this yields

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + k(q) = Bu,$$

where $F^e = Bu$, with $B = (I_m, 0_{n-m})^T$.

Dissipation can be added by extending $F = F^e + F^d$, with

$$F^d = -\frac{\partial D}{\partial \dot{q}}(\dot{q}),$$

where $D(\dot{q})$ is the Rayleigh dissipation function (content).

Hamiltonian systems

The Hamiltonian is given by $H(q,p) = T(q,p) + V(q)$, where T is the kinetic energy. The relation with the Lagrangian $\mathscr{L}$ is given by the Legendre transform, i.e.,

$$H(q,p) = p^T\dot{q} - \mathscr{L}(q,\dot{q}), \text{ with } p = \frac{\partial \mathscr{L}}{\partial \dot{q}}(q,\dot{q}) = M(q)\dot{q}.$$

The Hamiltonian system (without dissipation) is given by:

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q,p) \\ \dot{p} &= -\frac{\partial H}{\partial q}(q,p) + Bu. \end{aligned}$$

Linear systems

With $T(\dot{q}) = \frac{1}{2}\dot{q}^T M\dot{q} = \frac{1}{2}p^T M^{-1}p$, and $V(q) = \frac{1}{2}q^T Qq$, we have:

Euler-Lagrange equation:

$$M\ddot{q} + Qq = Bu$$

Hamiltonian system:

$$\begin{pmatrix}\dot{q}\\ \dot{p}\end{pmatrix} = \begin{pmatrix}0 & M^{-1}\\ -Q & 0\end{pmatrix}\begin{pmatrix}q\\ p\end{pmatrix} + \begin{pmatrix}0\\ B\end{pmatrix}u.$$

System response

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$$

Time signal	Laplace transform
$\delta(t)$	1
$1(t)$	$\frac{1}{s}$
$\frac{t^n}{n!}e^{at}1(t)$, $a \in \mathbb{C}$, $n \in \mathbb{N}$	$\frac{1}{(s-a)^{n+1}}$