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Introduction to Mathematical Finance (wi3417tu)
November 1st 2016, 18.30–21.30 uur

(No books, no notes.)

Please note: answers should be supplemented by motivation, explanation and/or calculation, whichever may be appropriate; in particular, whenever you apply a property of conditional expectations like TOWK, indicate this and explain why it applies. You may choose Dutch or English as the language to use for your answers. **Point distribution:** each *part* of a question is worth 1 point; the grade equals the number of points earned plus 1.

1. **a.** State the definition of arbitrage and show the following for the N-period binomial model: if, after discounting, every wealth process $X_0, \ldots, X_N$ is a martingale under $\tilde{\mathbb{P}}$, then there can be no arbitrage.

b. Consider a 1-period binomial model with parameters $S_0 = 2$, $S_1(H) = 5$, $S_1(T) = 1$, $r = \frac{1}{2}$. A derivative has payoff $V_1(H) = 2$, $V_1(T) = 10$. Construct a portfolio and use it to show that there is arbitrage if the time-zero price of the derivative is 3 Euros.

2. Consider the *floating strike lookback option* in the binomial model with parameters $S_0 = 12$, $u = 2$, $d = \frac{1}{2}$, $r = 0$, and $N = 3$. This option pays at expiration like a call option whose strike equals the lowest value the stock has attained; if we define $M_n = \min_{0 \le k \le n} S_k$, then the payoff can be expressed as: $V_N = S_N - M_N$.

a. Determine $V_1(H)$ and $V_1(T)$ by using the formula $V_n = \tilde{\mathbb{E}}_n \left[\frac{V_N}{(1+r)^{N-n}} \right]$.

b. Determine V_0 and the composition of the replicating portfolio (number of shares and bank balance) at $n = 0$.

c. Show that (S_n, M_n), $n = 0, 1, \ldots$ is a Markov process under $\tilde{\mathbb{P}}$.

3. Consider two probability measures $\mathbb{P}$ and $\tilde{\mathbb{P}}$ on the outcome space Ω. Every outcome $\omega \in \Omega$ has a positive probability under both probability measures. Let Z be the Radon-Nikodým derivative of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$, and Y an arbitrary random variable. Show that

$$\mathbb{E}[Y] = \tilde{\mathbb{E}}\left[\frac{1}{Z} \cdot Y\right].$$

4. Consider the binomial model with probabilities $p = 1/3$ and $q = 2/3$. Define $X_i = +2$ if $\omega_i = H$; $X_i = -1$ if $\omega_i = T$. Define $M_0 = 0$, $S_0 = 1$, and for $n \ge 1$:

$$M_n = \sum_{j=1}^{n} X_j, \qquad S_n = 2^{M_n}.$$

Each of the processes thus defined is *adapted.* You may use without proof that $M_0, M_1, \ldots$ is a Markov process. Below you are asked to answer questions on the properties of these processes; whether you answer "yes" or "no", give an argument to support your claim.

a. State the definition of a martingale and determine if $M_0, M_1, \ldots$ is a martingale.

b. Is $S_0, S_1, \ldots$ a martingale?

c. Because $M_0, M_1, \ldots$ is a Markov process and $S_n = h(M_n)$ with $h(m) = 2^m$, $S_0, S_1, \ldots$ is a Markov process as well. Is this true for any Markov process $M_0, M_1, \ldots$? Explain. For any function h?