Exam 23-6-2014 SDE and Solutions (Mastermath)

1. Let B_t be the standard Brownian motion, $B_T^* = \sup_{0\le t\le T} B_t$ and $\lambda > 0$.

   (a) Apply Doob's maximal inequality to prove that $\quad P(B_T^* \geq \lambda) \leq \frac{T}{\lambda^2}$. [3 pt]

   (b) Sharpen the inequality in (a) by applying the convex function $x \mapsto (x+c)^2$ to B_t for a suitable constant c to prove that $\quad P(B_T^* \geq \lambda) \leq \frac{T}{\lambda^2 + T}$. [2 pt]

Solution (a) Doob's maximal inequality says that for a continuous non-negative submartingale

$$\lambda P(M_T^* \geq \lambda) \leq E[M_T].$$

Now B_t is not non-negative, but we know that B_t^2 is a non-negative submartingale, since $f(x) = x^2$ is a convex function. Observe that $P(B_t^* \geq \lambda) \leq P((B_t^*)^2 \geq \lambda^2)$ and apply the inequality.

[*I have been lenient with the non-negativity, and only subtracted half a point in case you forgot.*]

(b) Now repeat the argument for the submartingale $(B_t + c)^2$ for a clever choice of c. Now $P(B_t^* \geq \lambda) \leq P((B_t + c)^2 \geq (\lambda + c)^2$ and apply Doob's maximum inequality again to find

$$P(B_t^* \geq \lambda) \leq \frac{E[(B_T + c)^2]}{(\lambda + c)^2} = \frac{T + c^2}{(\lambda + c)^2}$$

A clever choice of c minimizes this ratio: choose $c = T/\lambda$ and you find the desired inequality.

[*This exercise turned out to be very difficult; not many people solved it - but all women that participated in the exam were able to solve it! If you lost 1/2 point in (a) for not noticing negativity, I made that up with you if you said something about Jensen here. NB The reflection principle allows to compute $P(B_T^* \geq \lambda)$ in closed form!*]

2. (a) Use the Itô isometry to calculate the mean and the variance of $\int_0^t (B_s + s) dB_s$. [3 pt]

   (b) Calculate the mean and the variance of $\int_0^t (B_s + s) ds$. [4 pt]

Solution (a) Compare this exercise to equation (6.19) in the book! The Itô integral I_t is a martingale, so the mean is zero. Almost everybody got that. The Itô isometry says that

$$E[I_t^2] = E[\int_0^t (B_s + s)^2 ds]$$

where you need that the function $f(s, B_s) = s + B_s$ is in $\mathcal{H}^2$, but we know from the book that $f(s) = s$ and $f(B_s) = B_s$ are in $\mathcal{H}^2$, so the condition is satisfied. Now we apply Fubini and move the expectation inside, which is allowed since the integrand is non-negative $\int_0^t E((B_s + s)^2) ds = \int_0^t (s + s^2) ds = \frac{t^2}{2} + \frac{t^3}{3}$.

[*Almost everybody got mean zero (1 point) and was able to apply the Itô isometry (1 point). Some people failed to mention why Fubini applies or made a small mistake in the integral (costs you half a point). I was glad to see that most people knew their Fubini and some even checked it in detail.*]

(b) The mean follows from Fubini, though here you have to be a bit careful since B_t is not non-negative. You can notice for instance that $|B_t|$ is integrable. Fubini says that the mean is $E[I_t] = \int_0^t (E[B_s] + s)\, ds = \frac{t^2}{2}$.

To compute the variance, it is easiest to observe that the variance of $\int_0^t (B_s + s)ds$ is equal to the variance of $\int_0^t B_s ds$ since the other half of the integral is deterministic. This simplifies the computation since $\int_0^t B_s ds$ has mean zero. Now $E[(\int_0^t B_s ds)^2] = E[(\int_0^t B_s ds)(\int_0^t B_u du)] = E[\int_0^t \int_0^t B_s B_u ds du]$ and if we apply Fubini then we end up with $E[B_s B_u] = \min(s,u)$ and we have a standard integral which works out to $\frac{t^3}{3}$.

> *Officialy, you would have to check that this is allowed, for instance using that Brownian motion is square integrable. Steele glosses over this a bit and so I don't mind if you do the same, though some of you did check all the details!*
>
> *A few students were very clever and applied Itô's formula to solve this.*

Alternative way: Use Itô's formula to arrive at $tB_t = \int_0^t B_s ds + \int_0^t s dB_s$ so that the required quantity is the variance of $I_t = tB_t - \int_0^t s dB_s = \int_0^t (t-s)dB_s$. Then use Itô isometry

$$\mathrm{Var}(I_t) = E[(I_t^2)] = E\left[\int_0^t (t-s)^2 ds\right] = \frac{t^3}{3}.$$

> *Unfortunately, all these clever students made errors in their computation. Most importantly, they silently assumed that tB_t and $\int_0^t s dB_s$ are independent, which they are **not**, before applying Itô isometry. So perhaps it is not so clever to handle the exercise like this after all.*
>
> *I gave you 1 point for the mean, 2 points for splitting up the integral into a double dsdu integral or for being clever and moving on to the Itô isometry. And 1 point for finding the right answer. If you never mentioned Fubini, I deducted one point.*

3. Let B_t be the standard Brownian motion with respect to the filtration $\mathcal{F}_t$ and let τ_{-1} be the stopping time $\tau_{-1}(\omega) = \inf\{t\colon B_t(\omega) = -1\}$.

   (a) Show that $Z_t = B_{t\wedge\tau_{-1}}$ is a martingale with respect to $\mathcal{F}_t$. (refer to a theorem!) [2 pt]

   We introduce a time-change $X_t(\omega) = Z_{t/(1-t)}(\omega)$ for $0 \le t < 1$. It follows from exercise (a) that X_t is a martingale for $0 \le t < 1$ with respect to the time-change of the filtration.

   (b) Argue that $\lim_{t\to 1} X_t = -1$ with probability one. [1 pt]

   (c) We extend the definition by $X_t = -1$ for $t \ge 1$. Prove that the process X_t is not a martingale. [2 pt]

   (d) Prove that X_t is a local martingale. Use the localizing sequence τ_k that is defined by $\tau_k(\omega) = \inf\{t\colon X_t(\omega) = k\}$ if there exists such a t, or else $\tau_k(\omega) = k$. [2 pt]

Solution (a) This is a direct application of Doob's stopped martingale theorem: simply observe that B_t is a continuous martingale.

(b) Recall that $\tau_{-1} < \infty$ with probability one, see for instance p 56 of the book or remember your stochastic processes course. Once you have recalled this, you have solved the exercise.

(c) If X_t is a martingale then $E[X_t] = X_0 = 0$ which is obviously not true if $t \ge 1$.

> *Almost everybody was able to quote the correct theorems in (a) and (b) and use the expectation in (c).*

(d) Recall the definition of a local martingale. You have to show that $\tau_k \to \infty$ with probability one and that the stopped process $X_{t\wedge\tau_k}$ is a proper martingale.

To check that $\tau_k \to \infty$ recall that if you stop Brownian motion as soon as you reach k or -1, then this stopping time has $E[\tau] = k$ and you stop at k with probability $1/(k+1)$. So if $k \to \infty$ then you stop at -1 with probability one. You need to observe this to prove that $\tau_k \to \infty$.

Now you need to prove that the stopped process $X_{t\wedge\tau_k}$ is a martingale. Note that if $t \geq 1$ then $X_{t\wedge\tau_k}(\omega)$ is equal to -1 if $\tau_k(\omega) = k$ and otherwise $\tau_k(\omega) < 1$ so then the stopped process is equal to k. If $t < 1$ then X_t is equal to $Z_{t/(1-t}$ which is a stopped Brownian motion that is rescaled from time $(0, \infty)$ to time $(0, 1)$. In formula: $X_t = B_{t/(1-t)} \wedge \tau_{-1}/(1-\tau_{-1})$, but it is easier to write this as: $X_t = B_{s\wedge\tau_{-1}}$ with rescaled time $s = t/(1-t)$. So $X_{t\wedge\tau_k}$ is equal to $B_{s\wedge\tau_{-1}\wedge\tau_k}$ where by abuse of notation the stopping time τ_k for B_t has the standard meaning of stopping the Brownian motion at k.

Now collect all these observations: if $t < 1$ then $X_t \wedge \tau_k$ is equal to the Brownian motion B_s stopped at -1 or at k. If $t \geq 1$ then $X_{t\wedge\tau_k}$ is equal to the stopped Brownian motion for $s \to \infty$, which is either equal to -1 or to k. This is a martingale again by the stopped martingale theorem.

> *Part (d) was the hardest exercise. The main purpose of the exercise was to check whether you know the definition. Most of you got the definition right and were able to use the stopped martingale theorem. Hardly anybody checked that $\tau_k \to \infty$!*

4. Use the method of coefficient matching to solve the stochastic differential equation (SDE)

$$dX_t = -\frac{1}{2}X_t\, dt + \sqrt{1 - X_t^2}\, dB_t \quad \text{where } X_0 = 0.$$

Look for a solution of the form $u(B_t)$. [3 pt]

<u>*Remark:*</u> Note that the diffusion coefficient in the SDE, $\sigma(x) = \sqrt{1-x^2}$, is not Lipschitz in x near ± 1. So, the existence and uniqueness theorem do not "officially" apply here. This shows that the conditions in the theorem are *not necessary*. They are *sufficient conditions*. Once the "formal" answer by coefficient matching is obtained, you'll see that *if we wanted to* we could prove that our formal answer is an honest answer. We omit this work here, but sometimes it may be necessary.

Solution Let us look for a solution of the form $u(B_t)$ for the given SDE

$$dX_t = -\frac{1}{2}X_t\, dt + \sqrt{1 - X_t^2}\, dB_t, \qquad X_0 = 0. \tag{1}$$

Applying Itô formula to $u(B_t)$ we get

$$d[u(B_t)] = \frac{1}{2}u''(B_t)\, dt + u'(B_t)\, dB_t. \tag{2}$$

If $u(B_t)$ needs to be a solution to the SDE (1), we must have

$$d[u(B_t)] = -\frac{1}{2}u(B_t)\, dt + \sqrt{1 - u(B_t)^2}\, dB_t. \tag{3}$$

Matching the coefficients of the equations (2) and (3) we get

$$u'(x) = \sqrt{1 - u(x)^2} \quad \text{and} \quad u''(x) = -u(x).$$

This leads to the solution $u(x) = \sin(x + c)$, where c is a constant. The initial condition $X_0 = u(B_0) = u(0) = 0$, in turn, leads to $c = k\pi$, with integer k. Hence the solution to the given SDE (1) is

$$X_t = \sin(B_t + k\pi) = \sin(B_t), \quad t \geq 0.$$

5. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion on the filtered probability space $(\Omega, \mathcal{F}(\mathcal{F}_t), P)$. Show that $X_t = e^{\frac{1}{2}t}\cos B_t$ is an $\mathcal{F}_t$-martingale. [3 pt]

   [You may use any theorem, but make sure that the result is applicable by checking all the required conditions.]

Solution Consider the function $f(t,x) = e^{\frac{1}{2}t}\cos(x)$. Clearly $f(t,x)$ is differentiable in its first argument "t" with continuous (partial) derivative and twice differentiable w.r.t. its 2nd argument "x" with continuous (partial) derivative, i.e., $f \in C^{1,2}([0,T]\times\mathbb{R})$. Note that

$$f_x(t,x) = \frac{\partial f}{\partial x}(t,x) = -e^{\frac{1}{2}t}\sin(x) \quad \text{and} \quad f_{xx}(t,x) = -e^{\frac{1}{2}t}\cos(x).$$

Furthermore,

$$f_t(t,x) = \frac{\partial f}{\partial t}(t,x) = \frac{1}{2}e^{\frac{1}{2}t}\cos(x) = -\frac{1}{2}f_{xx}(t,x).$$

Applying the time-and-space variant of Itô formula we then have

$$\begin{aligned} dX_t &= d[f(t,B_t)] = f_x(t,B_t)dB_t + \underbrace{\left(f_t(t,B_t) + \frac{1}{2}f_{xx}(t,B_t)\right)}_{0} dt = f_x(t,B_t)dB_t \\ &= -e^{\frac{1}{2}t}\sin(B_t)dB_t. \end{aligned}$$

We can then rewrite X_t as

$$X_t = X_0 + \int_0^t dX_s = 1 + \int_0^t \left(-e^{\frac{1}{2}s}\sin(B_s)\right) dB_s.$$

Note that the integrand of the Itô integral on the right hand side (RHS) is bounded by $e^{\frac{1}{2}T}$ for every (t,ω) in $[0,T]\times\Omega$. Hence it is in $\mathcal{H}^2$. Hence the Itô integral on the RHS is a Martingale. Since addition of a constant does not change the martingale property, it is now proved that X_t is a Martingale.

6. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion on the filtered probability space $(\Omega, \mathcal{F}(\mathcal{F}_t), P)$. Suppose $(X_t)_{0\leq t\leq T}$ satisfies the stochastic differential equation (SDE)

$$\begin{aligned} dX_t &= \mu X_t\, dt + \sigma X_t\, dB_t, \quad 0 < t \leq T, \\ X_0 &= x_0, \end{aligned}$$

   and $(Y_t)_{0\leq t\leq T}$ evolves deterministically as

$$\begin{aligned} \dot{Y}_t &= rY_t, \quad 0 < t \leq T, \\ Y_0 &= y_0. \end{aligned}$$

   where μ, σ, r, x_0 and y_0 are positive constants, and μ is greater than r.

   (a) Use Itô formula to find the SDE satisfied by $\tilde{X}_t \equiv \dfrac{X_t}{Y_t}$, $0 \leq t \leq T$. [2 pt]

   (b) Using the Girsanov theorem, construct a probability measure under which $\tilde{X}_t$ is an $\mathcal{F}_t$-martingale. [3 pt]

Solution Let X_t and Y_t be as given in the question.

(a) Note that both X_t and Y_t are *standard* (or *Ito*) processes. [Actually, Y_t is trivially a standard process, because it is deterministic.] Observe that Y_t is always positive.

To obtain the SDE for $\tilde{X}_t \equiv \dfrac{X_t}{Y_t}$ we apply the Itô formula for two standard processes (in this case, *the product formula*) with $f(x,y) = \frac{x}{y}$, which is clearly in $C^{2,2}(\mathbb{R} \times (0,\infty))$. We then have (in box calculus notation)

$$\begin{aligned} d\tilde{X}_t &= d[f(X_t, Y_t)] \\ &= f_x dX_t + f_y dY_t + \frac{1}{2} f_{xx} dX_t \cdot dX_t + \frac{1}{2} f_{yy} dY_t \cdot dY_t + f_{xy} dX_t \cdot dY_t, \end{aligned} \tag{4}$$

where for brevity we have suppressed the arguments in the partial derivatives; for example, $f_x \equiv f_x(X_t, Y_t)$.

Note that, since Y_t is deterministic $dY_t \cdot dY_t = dX_t \cdot dY_t = 0$. Furthermore, we have

$$dX_t \cdot dX_t = \sigma^2 X_t^2 dt, \quad f_x(x,y) = \frac{1}{y}, \quad f_{xx}(x,y) = 0, \quad \text{and} \quad f_y(x,y) = -\frac{x}{y^2}.$$

The Itô product formula (4) then leads to the desired SDE

$$\begin{aligned} d\tilde{X}_t &= \frac{1}{Y_t} dX_t - \frac{X_t}{Y_t^2} dY_t = \frac{1}{Y_t} X_t [\mu\, dt + \sigma\, dB_t] - \frac{X_t}{Y_t^2} r Y_t\, dt \\ &= \tilde{X}_t [(\mu - r)\, dt + \sigma\, dB_t], \qquad (t > 0) \end{aligned} \tag{5}$$

with $\tilde{X}_0 = x_0/y_0$.

Alternative way: Note that $Y_t = y_0 e^{rt}$ and hence $\tilde{X}_t = y_0^{-1} e^{-rt} X_t$, apply time-space variant of Itô formula.

(b) From (5) we see that $\tilde{X}_t$ is also a standard process given by

$$\tilde{X}_t = \tilde{x} + \int_0^t \tilde{\mu}(\omega, s)\, ds + \int_0^t \tilde{\sigma}(\omega, s)\, dB_s,$$

where $\tilde{x} = \frac{x_0}{y_0}$, $\tilde{\mu}(\omega, s) = (\mu - r)\, \tilde{X}_s$, and $\tilde{\sigma}(\omega, s) = \sigma\, \tilde{X}_s$.

To turn $\tilde{X}_t$ into a martingale we necessarily need to remove the drift term. Note that $\theta(\omega, t) \equiv \dfrac{(\tilde{\mu}(\omega, t) - 0)}{\tilde{\sigma}(\omega, t)} = \dfrac{(\mu - r)}{\sigma}$ is a constant (and hence bounded). Hence we can apply the Girsanov theorem to obtain a new measure Q given by $\dfrac{dQ}{dP} = M_T$, where

$$M_t = e^{-\int_0^t \theta(\omega,s) dB_s - \frac{1}{2}\int_0^t \theta(\omega,s)^2 ds} = \exp\left(-\frac{(\mu - r)}{\sigma} B_t - \frac{1}{2} \frac{(\mu - r)^2}{\sigma^2} t \right)$$

so that under measure Q, $\tilde{X}_t$ can be expressed as

$$\tilde{B}_t = B_t + \int_0^t \theta(\omega, s) ds = B_t + \frac{(\mu - r)}{\sigma} t$$

is a Brownian motion and under Q,

$$\tilde{X}_t = \tilde{x} + \int_0^t \tilde{\sigma}(\omega, s)\, d\tilde{B}_s \quad \text{or equivalently} \quad d\tilde{X}_t = \tilde{\sigma}(\omega, t)\, d\tilde{B}_t = \sigma\, \tilde{X}_t\, d\tilde{B}_t.$$

We need to prove now that under Q, $\tilde{X}_t$ is a proper/honest martingale (instead of possibly being only a local martingale). Since $\tilde{B}_t$ is a Brownian motion under Q, the last SDE provides us the explicit expression for $\tilde{X}_t$, namely,

$$\tilde{X}_t = \tilde{x} \exp\left(\sigma\, \tilde{B}_t - \frac{1}{2} \sigma^2\, t \right), \qquad t \geq 0,$$

which is one of the three famous martingales related to the Brownian motion, multiplied by a constant (but that does not cause any problem).