SOLUTIONS EXAM EXERCISES

Solution 1. (i) We have

$$\begin{aligned} E|Z_t| &= EZ_t = e^{-\alpha t}\, E(e^{\sqrt{2\alpha}B_t}) = \frac{e^{-\alpha t}}{\sqrt{2\pi t}} \int_{\mathbb{R}} dx\, e^{\sqrt{2\alpha}x - \frac{1}{2t}x^2} \\ &= \frac{e^{-\alpha t + \alpha t}}{\sqrt{2\pi t}} \int_{\mathbb{R}} dx\, e^{-\frac{1}{2t}(x - t\sqrt{2\alpha})^2} = 1 < \infty \end{aligned}$$

and by independence of the increments

$$\begin{aligned} E(Z_t|\mathcal{F}_s) &= \exp^{-\alpha t} E\left(\exp(\sqrt{2\alpha}\,(B_t - B_s))\, \exp(\sqrt{2\alpha}\, B_s) \,|\, \mathcal{F}_s\right) \\ &= \exp^{-\alpha t} \exp(\sqrt{2\alpha} B_s)\, E\left(\exp(\sqrt{2\alpha}\,(B_{t-s}))\right) \end{aligned}$$

(ii)

$$\begin{aligned} Z_{\tau\wedge n} &= Z_{\tau\wedge n}\mathbf{1}_{(\tau\le n))} + Z_{\tau\wedge n}\mathbf{1}_{(\tau\ge n))} \\ &= Z_{\tau}\mathbf{1}_{(\tau\le n))} + Z_n\mathbf{1}_{(\tau\ge n))} \\ &= \exp(\sqrt{2\alpha}\, B_\tau - \alpha\tau)\mathbf{1}_{(\tau\le n))} + Z_n\,\mathbf{1}_{(\tau> n)} \\ &= \exp(\sqrt{2\alpha}\, a - \alpha\tau)\mathbf{1}_{(\tau\le n))} + Z_n\,\mathrm{P}(\tau > n). \end{aligned}$$

(iii) By $a, \alpha \ge 0$, we have $e^{\sqrt{2\alpha}B_n} \le e^{\sqrt{2\alpha}a}$ on $\{\tau \ge n\}$. Hence $Z_n \le e^{\sqrt{2\alpha}a}$ on $\{\tau > n\}$. Clearly $X \le e^{\sqrt{2\alpha}a}$ as well and hence

$$|Z_{\tau\wedge n}| = X\mathbf{1}_{(\tau\le n)} + Y_n\mathbf{1}_{(\tau\le n)} \le e^{\sqrt{2\alpha}a}.$$

(iv) By the assumption on the finiteness of the stopping time, we have that $Z_{\tau\wedge n} \to X$ almost surely. By (iii) we may apply dominated convergence to conclude that also

$$\lim_{n\to\infty} E(Z_{\tau\wedge n}) = E(X) = E(\exp(\sqrt{2\alpha}\, a - \alpha\tau)).$$

But by the stopping theorem, $E(Z_{\tau\wedge n}) = 1$ for all n. Hence

$$E(\exp(\sqrt{2\alpha}\, a - \alpha\tau)) = e^{\sqrt{2\alpha}\, a} E(e^{-\alpha\tau}) = 1.$$

That implies the statement.

(v) By Fubini and (iii)

$$\mathrm{E}\left[\frac{1}{\tau}\right] = \mathrm{E}\left[\int_0^\infty e^{-\lambda\tau}\, d\lambda\right] = \int_0^\infty \mathrm{E}\left[e^{-\lambda\tau}\right] d\lambda = \int_0^\infty e^{-\sqrt{2\lambda}a}\, d\lambda.$$

Substituting $u = \sqrt{2\lambda}a$ yields

$$du = \frac{a\,d\lambda}{\sqrt{2\lambda}} = \frac{a^2\,d\lambda}{u}$$

and hence

$$\int_0^\infty e^{-\sqrt{2\lambda}a}\,d\lambda = \frac{1}{a^2}\int_0^\infty e^{-u}u\,du = \frac{1}{a^2}.$$

(The last step either by partial integration or using the observation that the integral is the expectation of an exponential variable with parameter one.)

Solution 2. (i) by Ito formula with $f(x) = e^x$ and $X_t = \alpha B_t - \frac{1}{2}\alpha^2 t$, we have

$$dZ_t = e^{X_t}dX_t + \frac{1}{2}e^{X_t}dX_t dX_t = Z_t(\alpha db_t - \frac{1}{2}\alpha^2 dt) + \frac{1}{2}\alpha^2 dB_t dB_t = \alpha Z_t dB_t.$$

(ii) We have

$$\begin{aligned}
& \mathrm{E}\left[\int_0^t \left(\sum_{k=N}^\infty \alpha^k\, H_k(s,B_s)\right)^2 ds\right] \\
= & \int_0^t \left(\sum_{k,l=N}^\infty \alpha^{k+l}\,\mathrm{E}(H_k(s,B_s)H_l(s,B_s))\right)^2 ds = \int_0^t \left(\sum_{k=N}^\infty \frac{(\alpha^2 s)^k}{k!}\right) ds.
\end{aligned}$$

The statement follows now by the fact that the last line equals

$$e^{\alpha^2 x} - \sum_{k=0}^{N-1}\frac{(\alpha^2 t)^k}{k!}$$

and by the hint.

(iii) By Ito isometry we obtain from the assumption about the convergence of the generating function that

$$\begin{aligned}
& \lim_{N\to\infty} \mathrm{E}\left[\left(\int_0^t \exp\left(\alpha\,B_s - \frac{1}{2}\alpha^2 s\right) dB_s - \int_0^t \sum_{k=1}^N \alpha^k\, H_k(s,B_s)dB_s\right)^2\right] \\
= & \lim_{N\to\infty}\int_0^t \mathrm{E}\left[\left(\int_0^t \exp\left(\alpha\,B_s - \frac{1}{2}\alpha^2 s\right) - \sum_{k=1}^N \alpha^k\, H_k(s,B_s)\right)^2 ds\right] \\
= & \; 0.
\end{aligned}$$

The statement follows now by

$$\int_0^t \sum_{k=1}^N \alpha^k\, H_k(s,B_s)dB_s = \sum_{k=1}^N \alpha^k \int_0^t H_k(s,B_s)dB_s.$$

(iv) We have by (i) and (iii)

$$\begin{aligned}
& \sum_{k\geq 0} \alpha^k H_k(t, B_t) \\
= & \quad Z_t = \int_0^t dZ_s = \alpha \int_0^t Z_s \, dB_s \\
= & \quad \alpha \int_0^t \sum_{k\geq 0} \alpha^k H_k(s, B_s) \, dB_s = \sum_{k\geq 0} \alpha^{k+1} \int_0^t H_k(s, B_s) \, dB_s.
\end{aligned}$$

matching coefficients yields the statement.

Solution 3. (i) Using the ansatz

$$X_t = u(t) \left(x_0 + \int_0^t v(s) \, dB_s \right),$$

we obtain

$$\begin{aligned}
dX_t &= \left(x_0 + \int_0^t v(s) \, dB_s \right) du(t) + u(t) \, d \left(x_0 + \int_0^t v(s) \, dB_s \right) \\
&\quad + du \, d \left(x_0 + \int_0^t v(s) \, dB_s \right) \\
&= \dot{u} \left(x_0 + \int_0^t v(s) \, dB_s \right) dt + uv \, dB_s.
\end{aligned}$$

Matching coefficients yields

$$\dot{u} = -au, \quad v = \sigma/u.$$

That implies

$$u(t) = Ae^{-at}, \quad v = \tfrac{\sigma}{A} e^{at}.$$

Hence

$$X_t = Ae^{-at} \left(x_0 + \int_0^t \frac{\sigma}{A} e^{as} \, dB_s \right)$$

and by adjusting the free parameter A to the requirement $X_0 = x_0$ we finally obtain

$$X_t = e^{-at} \left(x_0 + \sigma \int_0^t e^{as} \, dB_s \right).$$

(ii) In the sequel, we use without proof the fact mentioned in the lecture that processes of the form

$$Y_t = \int_0^t f(s) dB_s$$

with deterministic f are Gaussian with *independent increments*. Hence (assume without loss of generality $t \geq s$)

$$E(Y_s Y_t) = E(Y_s^2) + E(Y_s(Y_t - Y_s)) = E(Y_s^2)$$

and we can apply this letting

$$Y_t = \sigma \int_0^t e^{as}\, dB_s$$

in the following way (again $s \leq t$)

$$\begin{aligned}
E(X_s X_t) &= E(e^{-a(s+t)}X_0^2 + X_0 e^{-as} Y_t + X_0 e^{-at} Y_s + e^{-a(s+t)} Y_s Y_t) \\
&= e^{-a(s+t)} E(X_0^2) + e^{-a(s+t)} E(Y_s^2) \\
&= e^{-a(s+t)} \left(E(X_0^2) + \sigma^2 \, E((\int_0^s e^{au} dB_u)^2)) \right) \\
&= e^{-a(s+t)} \left(E(X_0^2) + \sigma^2 \, E(\int_0^s e^{2au} du) \right) \\
&= e^{-a(s+t)} \left(\frac{\sigma^2}{2a} + \sigma^2 \, \frac{e^{2as} - 1}{2a} \right) \\
&= e^{-a(t-s)} \frac{\sigma^2}{2a}.
\end{aligned}$$

The same calculation with $s \geq t$ yields the statement.

Solution 4. Let

$$\mathcal{G} := \{F \in \mathcal{F}_\infty \, : \, \forall_{\varepsilon > 0} \exists_{n \geq 0, g \in \mathcal{F}_n} \, : \, P(F \triangle G) < \epsilon\}.$$

(i) We want to show that $\mathcal{G} = \mathcal{F}$. First we show that $\mathcal{G}$ is a σ-algebra.

(a) $\emptyset \subset \mathcal{G}$: For $F = \emptyset$ let $n = 0$ and $G = \emptyset \in \mathcal{F}_0$. Then $G \triangle F = \emptyset$ and also $P(G \triangle F) = 0 < \epsilon$.

(b) $F \in \mathcal{G} \Rightarrow F^c \in \mathcal{G}$: Assume $F \in \mathcal{G}$. Then there is some $n \geq 0$ and $G \in \mathcal{F}_n$ such that $P(G \triangle F) < \epsilon$. But $G \in \mathcal{F}_n$ implies $G^c \in \mathcal{F}_n$. By $G \triangle F = G^c \triangle F^c$ we obtain $P(G^c \triangle F^c) = P(G \triangle F) < \epsilon$ and that implies the statement.

(c) $F_k \in \mathcal{G}, k \geq 1 \Rightarrow \bigcup_{k \geq 1} F_k \in \mathcal{G}$: First of all, note that

$$\begin{aligned}
(F_1 \cup F_2) - (G_1 \cup G_2) &= (F_1 - (G_1 \cup G_2)) \cup (F_2 - (G_1 \cup G_2)) \\
&\subseteq (F_1 - G_1) \cup (F_2 - G_2)
\end{aligned}$$

implies that

$$(F_1 \cup F_2) \triangle (G_1 \cup G_2) \subseteq (F_1 \triangle G_1) \cup (F_2 \triangle G_2).$$

That implies that all finite unions of sets in $\mathcal{G}$ are contained in $\mathcal{G}$. (For unions of two sets, you can see this by choosing G_1 and G_2 such that $P(F_1 \triangle G_1) = P(F_2 \triangle G_2) = \epsilon/2$.) Now we reduce the problem (c) to unions of finitely many sets: Let $a_r := P(\bigcup_{k=1,\ldots,r} F_k)$. Then a_k is monotonously increasing and bounded above by 1. Hence a_k converges and we can therefore choose some $K > 0$ such that

$$P(\bigcup_{k \geq K+1} F_k - \bigcup_{k=1,\ldots,K} F_k) \leq \epsilon/2.$$

For each $k \leq K$ we now have by definition some $G_k \in \mathcal{F}_{n_k}$ with $P(F_k \triangle G_k) < \frac{\epsilon}{2K}$. Let $N := \max_{k \leq K} n_k$. Then $\cup_{k=1,...,K} G_k \in \mathcal{F}_N$ and we obtain by the formula above

$$\begin{aligned}
& P(\bigcup_{k\geq 1} F_k \triangle \bigcup_{k=1,\ldots,K} G_k) \\
= \; & P(((\bigcup_{k\geq K+1} F_k - \bigcup_{k=1,\ldots,K} F_k) \cup \bigcup_{k=1,\ldots,K} F_k) \triangle \bigcup_{k=1,\ldots,K} G_k) \\
\leq \; & P(\bigcup_{k=1,\ldots,K} F_k \triangle \bigcup_{k=1,\ldots,K} G_k) + P(\bigcup_{k\geq K+1} F_k - \bigcup_{k=1,\ldots,K} F_k) \\
= \; & \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

Thus, $\mathcal{G}$ is a σ-algebra.

It remains to show that $\mathcal{F}_\infty \subseteq \mathcal{G}$, since by definition we already know that $\mathcal{G} \subseteq \mathcal{F}_\infty$. Because $\mathcal{F}_\infty$ is the *smallest* σ-algebra containing all $\mathcal{F}_n$, we are done if we can prove that $\mathcal{G}$ contains as well all $\mathcal{F}_n$. Let thus $F \in \mathcal{F}_n$. Then we may choose $G = F \in \mathcal{F}_n$ and obtain $P(F \triangle G) = P(F \triangle F) = P(\emptyset) = 0$. Hence $F \in \mathcal{G}$ and thus $\mathcal{F}_n \subseteq \mathcal{G}$. Since n was arbitrary, that implies the statement.

(ii) Let $\xi = 1_F$, $F \in \mathcal{F}_\infty$ an indicator function. Then the statement follows from $E|1_F - 1_G|^p = P(F \triangle G)$ and (i). Now step functions are dense in $L^p(\Omega, \mathcal{F}_\infty)$ and we thus have always some step function $s := \sum_{i=1}^n f_i \, 1_{F_i}$ such that

$$E|\xi - \sum_{i=1}^n f_i \, 1_{F_i}|^p < \epsilon/2.$$

Now we use that by (i) we can find $G_i \in \mathcal{F}_{n_i}$ such that for all $i = 1, ..., n$ we have

$$P(F_i \triangle G_i) < \frac{(\epsilon/2)^{1/p}}{n|f_i|}.$$

Thus, the step function $\eta := \sum_{i=1}^n f_i \, 1_{G_i}$ is $\mathcal{F}_N$-measurable with $N := \max n_i$ and we have

$$\begin{aligned}
E|\xi - \eta|^p \; & \leq \; E|\xi - s|^p + E|\eta - s|^p < \epsilon/2 + E|\sum_{i=1}^n f_i \, (1_{F_i} - 1_{G_i})|^p \\
& \leq \; \epsilon/2 + \left(\sum_{i=1}^n |f_i| \, P(F_i \triangle G_i) \right)^p < \epsilon/2 + ((\epsilon/2)^{1/p} \sum_{i=1}^n \frac{1}{n})^p = \epsilon.
\end{aligned}$$

That implies the statement.

(iii) Without loss of generality we assume that ξ is $\mathcal{F}_\infty$-measurable (otherwise we substitute ξ by $\hat{\xi} := E(\xi \,|\, \mathcal{F}_\infty)$. Let now $\eta \in L^p(\Omega, \mathcal{F}_n)$ be a function such that $E|\xi - \eta|^p < \epsilon/2$. Then by triangle inequality and L^p-contractivity of conditional expectation

$$\begin{aligned}
E|E(\xi \,|\, \mathcal{F}_m) - \xi|^p \; & \leq \; E|E(\xi \,|\, \mathcal{F}_m) - \eta|^p + E|\eta - \xi|^p \\
& = \; E|E(\xi - \eta \,|\, \mathcal{F}_m)|^p + E|\eta - \xi|^p \leq \epsilon/2 + \epsilon/2 = \epsilon
\end{aligned}$$

for all $m \geq n$. Since we can find some n and η for all values of $\epsilon > 0$ by (ii), that implies the statement.

(iv) Conditional expectation is only defined almost surely. Hence it only makes sense to ask for pointwise convergence almost surely. But actually this is provided by the *martingale convergence theorem*. The first observation is that for $\xi \in L^p(\Omega, \mathcal{F})$, by contractivity of conditional expectation $E|E(\xi \,|\, \mathcal{F}_n)|^p < \infty$ for all n. Hence

$$X_n := E(\xi \,|\, \mathcal{F}_n)$$

is an L^p-martingale by *tower property*

$$E(X_{n+1} \,|\, \mathcal{F}_n) = E(E(\xi \,|\, \mathcal{F}_{n+1}) \,|\, \mathcal{F}_n) = E(\xi \,|\, \mathcal{F}_n) = X_n.$$

Thus, by *martingale convergence*, there is some random variable X_∞ to which X_n converges almost surely and in L^p. By (iii), this random variable must almost surely coincide with $E(\xi \,|\, \mathcal{F}_\infty)$.