Stochastic Differential Equations (3TU),
12 June, 2006, 14.00 - 17.00

Grading: $(1+1)+(1)+(1+1)+(1+1+1+1)+(1)+(1)$.

1. Let $\{X_i : 1 \le i < \infty\}$ be a sequence of independent random variables with the probability distribution given by

$$P(X_i = 1) = P(X_i = -1) = \frac{1}{2}.$$

Let S_n denote the partial sums

$$S_n = X_1 + \ldots + X_n, \quad n \ge 1,$$

and let $S_0 \equiv 0$. Define

$$Y_n = (-1)^n \cos\left[\pi(S_n + A)\right], \quad n \ge 1,$$

where A is a given positive integer. Let

$$\tau = \min\{Y_n : S_n = A \text{ or } S_n = -A\}.$$

   (a) Prove that $\{Y_n : 1 \le n < \infty\}$ is a martingale with respect to the sequence $\{X_i : 1 \le i < \infty\}$.

   (b) Show that τ is a stopping time and that $P(\tau < \infty) = 1$.
   Hint: let E_k denote the event that $X_i = 1$ for all integers $i = 2kA, \ldots, 2(k+1)A - 1$. Then $\{\tau > 2nA\} \subset \bigcap_{k=0}^{n-1} E_k^c$ (explain why).

2. Let $\{B_t : 0 \le t < 1\}$ be a Standard Brownian Motion on $[0, 1)$. Define

$$X_t = \frac{1}{\sqrt{a}} B_{at}, \quad 0 \le t < \frac{1}{a},$$

where $a > 0$ is a given constant. Show that $\{X_t : 0 \le t < 1/a\}$ is a Standard Brownian Motion on $[0, 1/a)$.

3. Let $\{B_t : t \ge 0\}$ be a Standard Brownian Motion and let $\{\phi_k\}$ be a complete, orthonormal basis in $L_2[0,T]$, i.e.

$$\langle \phi_k, \phi_l \rangle = \int_0^T \phi_k(t)\phi_l(t)\,dt = \begin{cases} 1, & k = l \\ 0, & k \ne l, \end{cases}$$

and, for any $\phi \in L_2[0,T]$,

$$\lim_{N\to\infty} \|\phi - \sum_{k=1}^{N} \langle \phi, \phi_k \rangle \phi_k\|^2 = 0,$$

where $\|.\|$ denotes the norm in $L_2[0,T]$.

(a) Show that

$$E\left[\left(\int_0^T \phi(t)\,dB_t - \sum_{k=1}^{N}\langle\phi,\phi_k\rangle\int_0^T \phi_k(t)\,dB_t\right)^2\right] = \|\phi - \sum_{k=1}^{N}\langle\phi,\phi_k\rangle\phi_k\|^2.$$

(b) Show that

$$\int_0^T \phi(t)\,dB_t = \sum_{k=1}^{\infty}\langle\phi,\phi_k\rangle\int_0^T \phi_k(t)\,dB_t,$$

where the infinite sum in the right hand side converges in the mean-square sense.

4. Let $\{B_t : t \geq 0\}$ be a Standard Brownian Motion. Define

$$X_t = \exp(B_t - t/2), \quad t \geq 0.$$

(a) Show that X_t is a martingale with $E(|X_t|) = 1$ for all $t \geq 0$.

(b) Show that X_t converges *in probability* to $X_\infty \equiv 0$ as $t \to \infty$.

(c) Give the formulation of the Martingale Convergence Theorem in continuous time. Explain carefully that, as $t \to \infty$, X_t converges to $X_\infty \equiv 0$ *with probability 1*, i.e. $P(\lim_{t\to\infty} X_t = X_\infty) = 1$.
Discuss $\lim_{t\to\infty} \|X_t - X_\infty\|_p$ for $p \geq 1$.

(d) Give the definition of uniform integrability. Show that X_t is not uniformly integrable.

5. Let $\{B_t : t \geq 0\}$ be a Standard Brownian Motion. Show that

$$dB_t^n = \frac{1}{2}n(n-1)B_t^{n-2}\,dt + nB_t^{n-1}\,dB_t.$$

6. Use appropriate coefficient matching to solve the SDE

$$dX_t = \mu\,dt + \sigma X_t\,dB_t \text{ with } X_0 = 0.$$