TU Delft | Mekelweg 4
Faculteit EWI, DIAM | 2628 CD Delft

Exam Applied Functional Analysis
January 25, 2017, 13.30 - 16.30

All answers should be carefully motivated.

Results from the course book and notes may be used without proof, provided they are cited correctly.

The use of any electronic equipment is prohibited.

Grades: $\frac{1}{3}\big[(3+2)+(3+3)+(2+3)+(1+3+2)+(3+2)+(3\ \texttt{free})\big]$

Unless otherwise stated, the scalar field $\mathbb{K}$ can be both $\mathbb{R}$ and $\mathbb{C}$.

1. Let Y be a linear subspace of a Banach space X, let Z be a finite-dimensional Banach space, and let $T : Y \to Z$ be a bounded linear operator.

   (a) Show that ϕ has an extension to a bounded operator $\widetilde{T} : X \to Z$ of norm $\|\widetilde{T}\| \leqslant n\|T\|$.
   Hint: Z is isomorphic to $\mathbb{K}^n$ for some non-negative integer n.

   (b) Is $\widetilde{T}$ compact? If yes, explain why; if no, provide a counterexample.

2. On the Hilbert space $H = L^2(0,1)$ define the linear operator $T : H \to H$ by
$$Tf(t) := \frac{1}{\sqrt{t}} \int_0^t f(s)\,\mathrm{d}s \quad (t \in (0,1)).$$

   (a) Show that T is well defined (i.e., show that $Tf \in H$ for all $f \in H$) and bounded.

   (b) Find an expression for the Hilbert space adjoint $T^\star$.

3. Let U be a unitary operator acting on a complex Hilbert space H.

   (a) Show that for all non-zero $\lambda \in \{z \in \mathbb{C} : |z| \leqslant 1\}$ the following equivalence holds for the spectra of U and its inverse U^{-1}: we have $\lambda \in \sigma(U)$ if and only if $1/\lambda \in \sigma(U^{-1})$.
   Hint: To do some educated guesswork, rewrite the function $1/(\frac{1}{\lambda} - \frac{1}{z})$.

(b) Using this, prove that $\sigma(U) \subseteq \{z \in \mathbb{C} : |z| = 1\}$.

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4. Let $1 \leqslant p \leqslant \infty$ be fixed, and denote by ℓ^p the Banach space of all p-summable scalar sequences $c = (c_n)_{n=1}^{\infty}$.

   (a) Show that for all $t \geqslant 0$ the linear operator S_t defined by

   $$S_t : (c_n)_{n=1}^{\infty} \mapsto (e^{-nt} c_n)_{n=1}^{\infty}$$

   is bounded on ℓ^p.

   (b) Show that the family $(S_t)_{t \geqslant 0}$ defines a C_0-semigroup on ℓ^p if and only if $1 \leqslant p < \infty$.

   (c) For $1 \leqslant p < \infty$, show that its generator $(A, \mathsf{D}(A))$ is given by

   $$\begin{aligned} \mathsf{D}(A) &= \{c = (c_n)_{n=1}^{\infty} \in \ell^p : \ (nc_n)_{n=1}^{\infty} \in \ell^p\}, \\ Ac &= -(nc_n)_{n=1}^{\infty}. \end{aligned}$$

5. For a function $f \in W^{1,1}(0,1)$, define $g \in L^1_{\text{loc}}(-1,1)$ by $g(x) := f(|x|)$.

   (a) Show that $g \in W^{1,1}(-1,1)$.
   Hint: Use that f has a version belonging to $C[0,1]$.

   (b) Show, by way of example, that for $f \in W^{2,1}(0,1)$ it may happen that $g \notin W^{2,1}(-1,1)$.

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