TU Delft Mekelweg 4
Faculteit EWI, DIAM 2628 CD Delft

Exam Applied Functional Analysis
January 20, 2016, 13.30 - 16.30

All answers should be carefully motivated.

Results from the course book and notes may be used without proof, provided they are cited correctly.

The use of any electronic equipment is prohibited.

Grading: $\frac{1}{3} \times [(3+2)+(3+2+3)+(2+2+2+2)+(2+2+2)+(3 \texttt{ free})]$

Unless otherwise stated, the scalar field $\mathbb{K}$ can be both $\mathbb{R}$ and $\mathbb{C}$.

1. Let X_0 be a dense subspace of a Banach space X, let Y be a Banach space, and let $T_0 : X_0 \to Y$ be a linear operator.

   (a) Prove that if T_0 is bounded, then T_0 can be extended to a bounded linear operator $T : X \to Y$ and show that this extension is unique.
   Solution: A complete proof is in the Notes. (Correct def. of the extension: 1 pt. Well-definedness: 1/2 pt. Boundedness of the extension: 1/2 pt. Uniqueness: 1 pt.)

   (b) Give an example of a linear operator $T_0 : X_0 \to Y$ that cannot be extended to a bounded operator $T : X \to Y$. Motivate your answer.
   Solution: Take for instance $X = Y = C[0,1]$, $X_0 = C^1[0,1]$, and $T_0 f := f'$, the derivative operator. It it had a bounded extension, then T_0 would be bounded with respect to the supremum norm on $C^1[0,1]$. It is, however, easy to produce functions $f_n \in C^1[0,1]$ satisfying $\|f_n\|_\infty = 1$ and $\|f_n'\| \geqslant n$ (just make sure that the slope gets steep enough at some point).

2. Consider the linear mapping $T : L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$ defined by

$$Tf(\xi) := \int_{-\infty}^{\infty} f(x)\, \sin(\xi x)\, dx \quad (f \in L^1(\mathbb{R}),\ \xi \in \mathbb{R}).$$

   (a) Show that T is bounded and compute its norm.
   Solution: From

$$\|Tf(\xi)\|_\infty \leqslant \int_{\mathbb{R}} |f(x)\sin(\xi x)|\, dx \leqslant \int_{\mathbb{R}} |f(x)|\, dx = \|f\|_1$$

we see, after taking the sup over ξ, that T is bounded and $\|T\| \leqslant 1$ (1 1/2 pt). To see that also $\|T\| \geqslant 1$, fix $\varepsilon > 0$. Choose $\delta > 0$ so small that $\sin(x) \geqslant 1-\varepsilon$ for all $x \in [\frac{1}{2}\pi, \frac{1}{2}+\delta]$. Set $f := \frac{1}{\delta}1_{(\frac{1}{2}\pi, \frac{1}{2}\pi+\delta)}$. This function satisfies $\|f\|_1 = 1$. Moreover,

$$\begin{aligned}\|T\| \geqslant \|Tf\| \geqslant |Tf(1)| &= \frac{1}{\delta}\Big|\int_{\frac{1}{2}\pi}^{\frac{1}{2}\pi+\delta} \sin(x)\,dx\Big| \\ &\geqslant \frac{1}{\delta}\Big|\int_{\frac{1}{2}\pi}^{\frac{1}{2}\pi+\delta} (1-\varepsilon)\,dx\Big| = 1-\varepsilon.\end{aligned}$$

It follows that $\|T\| \geqslant 1$ (1 1/2 pt). Since we had already that $\|T\| \leqslant 1$, we conclude that $\|T\| = 1$.

(b) Show that the space $C_0(\mathbb{R})$ consisting of all continuous functions $f : \mathbb{R} \to \mathbb{K}$ satisfying $\lim_{|x|\to\infty} f(x) = 0$, endowed with the supremum norm, is a closed subspace of $L^\infty(\mathbb{R})$.

Solution: The inclusion is given by identifying a continuous function with its equivalence class modulo null sets. Since the essential supremum (as used in the definition of $L^\infty(\mathbb{R})$ agrees with the supremum for continuous functions, this identification is isometric (1/2 pt). To show that $C_0(\mathbb{R})$ is a closed subspace, let $f_n \to f$ in $L^\infty(\mathbb{R})$ with $f_n \in C_0(\mathbb{R})$ (1/2 pt). Then f is continuous, it being the uniform limit of a sequence of continuous functions (1/2 pt). Let $\varepsilon > 0$. Choose N so large that $\|f_n - f\|_\infty < \varepsilon$ for all $n \geqslant N$. Choose $R > 0$ so large that $|f_N(x)| \geqslant \varepsilon$ for $|x| \geqslant R$. Then, for $|x| \geqslant R$,

$$|f(x)| \leqslant |f_N(x)| + |f(x) - f_N(x)| < 2\varepsilon.$$

This proves that $\lim_{|x|\to\infty} |f(x)| = 0$, i.e. $f \in C_0(\mathbb{R})$ (1/2 pt).

(c) Show that T maps $L^1(\mathbb{R})$ into $C_0(\mathbb{R})$.

Hint: By direct computation, check that T maps indicator functions $1_{(a,b)}$ into $C_0(\mathbb{R})$. Combine this with the fact (which may be used without proof) that step functions are dense in $L^1(\mathbb{R})$. Use the result of Problem 1.

Solution: Fist of all, by dominated convergence we see that Tf is sequentially continuous, hence continuous, for all $f \in L^1(\mathbb{R})$ (use $|f|$ as majorising function) (1 pt). Following the hint we compute (1/2 pt)

$$T1_{(a,b)}(\xi) = \int_a^b \sin(\xi x)\,dx = \frac{1}{\xi}(\cos(\xi a) - \cos(\xi b))$$

which decays to 0 as $t \to \infty$, showing that $T1_{(a,b)} \in C_0(\mathbb{R})$ (1/2 pt). By taking linear combinations, it follows that $Tf \in C_0(\mathbb{R})$ for all step functions. By Problem one, T has an extension to a bounded operator, which we call $\overline{T}$ for the moment, from $L^1(\mathbb{R})$ to $C_0(\mathbb{R})$ (1/2 pt). To complete the proof we must show that this extension agrees with the Fourier transform. To this end let $f \in L^1(\mathbb{R})$ and choose step functions $f_n \to f$ in $L^1(\mathbb{R})$. Then, by the boundedness of T and $\overline{T}$,

$$\overline{T}f = \lim_{n\to\infty} \overline{T}f_n = \lim_{n\to\infty} Tf_n = Tf$$

where we view $\overline{T}f$ as a function in $C_0(\mathbb{R})$ and Tf as a function in $L^\infty(\mathbb{R})$. It follows that $\overline{T}f = Tf$ as functions in $L^\infty(\mathbb{R})$ (1/2 pt).

3. Let H be a Hilbert space with inner product $(\cdot|\cdot)$ and orthonormal basis $(h_n)_{n\geqslant 1}$. A bounded operator $T \in \mathscr{L}(H)$ is said to be a *Hilbert-Schmidt operator* if

$$\|T\|_{\mathscr{L}_2(H)} := \Big(\sum_{n\geqslant 1} \|Th_n\|^2\Big)^{1/2} < \infty.$$

(a) Show that if $(h'_n)_{n\geqslant 1}$ is another orthonormal basis of H, then

$$\sum_{n\geqslant 1} \|Th'_n\|^2 = \sum_{n\geqslant 1} \|Th_n\|^2,$$

i.e., the definition of a Hilbert-Schmidt operator does not depend on the choice of the orthonormal basis.

Solution: We have

$$\sum_n \|Th'_n\|^2 = \sum_n \sum_m (Th'_n|h_m) = \sum_m \sum_n (h'_n|T^\star h_m) = \sum_m \|T^\star h_m\|^2.$$

where the change of summation order is justified by absolute convergence of the double sum. By the same reasoning (with $h'_n = h_n$), $\sum_n \|Th_n\|^2 = \sum_m \|T^\star h_m\|^2$. Comparing these identities gives the result.

(b) Show that if T is a Hilbert-Schmidt operator, then $\|T\| \leqslant \|T\|_{\mathscr{L}_2(H)}$.

Solution: Fix any h of norm one and compute the HS norm using any orthonormal basis starting with h, say $(h_n)_{n\geqslant 1}$ with $h = h_1$. Then

$$\|Th\|^2 = \|Th_1\|^2 \leqslant \sum_{n\geqslant 1} \|Th_n\|^2 = \|T\|_2^2.$$

Taking the supremum over all h of norm one gives $\|T\| \leqslant \|T\|_2$.

(c) Show that the set $\mathscr{L}_2(H)$ of all Hilbert-Schmidt operators on H is a Hilbert space with respect to the inner product

$$(T|S)_{\mathscr{L}_2(H)} := \sum_{n\geqslant 1} (Th_n|Sh_n).$$

Solution: The properties of an inner product are routinely verified (you should have done this in the exam, 1/2 pt)) and here we establish completeness. Suppose $(T_n)_{n\geqslant 1}$ is a Cauchy sequence with respect to the HS norm. From $\|\cdot\|_{\mathscr{L}(H)} \leqslant \|\cdot\|_{\mathscr{L}_2}(T)$ we see that $(T_n)_{n\geqslant 1}$ is Cauchy in $\mathscr{L}(H)$ and therefore converges to some $T \in \mathscr{L}(H)$ (1/2 pt). To show that T is HS, fix $\varepsilon > 0$ and choose N so large that $\|T_n - T_m\|_2 \leqslant \varepsilon$ for $n, m \geqslant N$. For any integer K and all $n, m \geqslant N$,

$$\sum_{k=1}^{K} \|(T_n - T_m)h_k\|^2 \leqslant \|T_n - T_m\|_2^2 \leqslant \varepsilon^2.$$

Passing to the limit $m \to \infty$, this gives

$$\Big\| \sum_{k=1}^{K} (T_n - T)h_k\|^2 \leqslant \varepsilon^2.$$

Passing to the limit $K \to \infty$, this gives $T_n - T \in \mathscr{L}_2(H)$ and $\|T_n - T\|_2 \leqslant \varepsilon$. But then also $T = T_n - (T_n - T) \in \mathscr{L}_2(H)$, and the previous estimate means that $T_n \to T$ in $\mathscr{L}_2(H)$ (1 pt).

(d) Show that every Hilbert-Schmidt operator is compact.

Hint: Consider $T_n = T \circ P_n$, where P_n is the orthogonal projection onto the span of $\{h_1, \dots, h_n\}$.

Following the hint, note that P_n has finite-dimensional range and therefore is compact. Hence, TP_n is compact (1 pt). To finish the proof, it remains to show that $\|T_n - T\| \to 0$, for then T is compact being a uniform limit of compact operators (1/2 pt).

Now

$$\|T_n - T\|^2 \leqslant \|T_n - T\|_2^2 = \sum_{j=1}^{\infty} \|(T - TP_n)h_n\|^2 = \sum_{j=n+1}^{\infty} \|Th_n\|^2,$$

and this tends to 0 as $n \to \infty$ (1 pt).

(e) Let $H = \mathbb{K}^d$ with the standard unit vector basis and let $A = (a_{ij})_{i,j=1}^{d}$ be a $(d \times d)$ matrix with coefficients in $\mathbb{K}$. Show that A, considered as a bounded operator on H, is a Hilbert-Schmidt operator and express $\|A\|_{\mathscr{L}_2(\mathbb{K}^d)}$ in terms of the coefficients a_{ij}.

Solution: Take the standard unit vectors e_n (1/2 pt). Then Ae_n is the n-th column vector of A (1/2 pt). Accordingly (1 pt),

$$\|A\|_2^2 = \sum_{n=1}^{d} \|Ae_n\|^2 = \sum_{n=1}^{d} \sum_{m=1}^{d} |a_{mn}|^2.$$

4. In this problem we work over the real scalar field. Consider the boundary value problem

$$u'' - qu = -f, \quad u(0) = u(1) = 0,$$

where $f \in L^2(0,1)$ is a given function and $q : [0,1] \to \mathbb{R}$ is a non-negative continuous function. We call a function $u \in H_0^1(0,1)$ a *weak solution* if

$$\int_0^1 u'(x)\phi'(x)\,dx + \int_0^1 q(x)u(x)\phi(x)\,dx = \int_0^1 f(x)\phi(x)\,dx \quad \forall \phi \in C_c^1(0,1).$$

(a) Show that

$$\varphi(g) := \int_0^1 f(x)g(x)\,dx \quad (g \in H_0^1(0,1))$$

defines a bounded linear functional on $H_0^1(0,1)$.

(b) Show that

$$a(g_1, g_2) := \int_0^1 g_1'(x)g_2'(x) + q(x)g_1(x)g_2(x)\,dx \quad (g_1, g_2 \in H_0^1(0,1))$$

defines a continuous coercive bilinear form on $H_0^1(0,1)$.

(c) Prove that the above boundary value problem has a unique weak solution.

Solution: A more general case is in the Notes. For instance, coercivity follows from

$$a(g, g) \geqslant \|g'\|_2^2 \geqslant C^{-2}\|g\|_{H^1}^2,$$

where C is the constant of the Poincaré inequality (in the format $\|g\|_{H^1} \leqslant C\|g'\|_2$ for $g \in H_0^1$).

`-- The end --`