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Assignment exam Applied Functional Analysis
January 8, 2016, 10.45 - 12.30

Solutions should be given in full detail

Grading: $3+(1+1\frac{1}{2}+1\frac{1}{2})+3$

All vector spaces are over the scalar field $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$.

1. Let $(x_n)_{n\geqslant 1}$ be a sequence in a Hilbert space H with inner product $(\cdot|\cdot)$ and suppose there exists an $x\in H$ such that $(x_n|y)\to(x|y)$ for all $y\in H$ and $\|x_n\|\to\|x\|$. Show that $x_n\to x$ in H.

   Solution: Was given in the assignment class.

2. Let X and Y be Banach spaces, and let $D(A)$ be a linear subspace of X. Let $A: D(A)\to Y$ be a linear operator. In this context the subspace $D(A)$ is called the *domain* of A. Such an operator is said to be *closed* if its graph $\{(x,Ax): x\in D(A)\}$ is closed in $X\times Y$.

   (a) Show that the following assertions are equivalent:

   (i) A is closed;

   (ii) whenever $(x_n)_{n\geqslant 1}$ is a sequence in $D(A)$ and x and y are elements in X and Y respectively such that $x_n\to x$ and $Ax_n\to y$, then $x\in D(A)$ and $Ax=y$.

   (b) Let $X=Y=C[a,b]$, and consider the linear operator $A: D(A)\to Y$ with domain $D(A)=C^1[a,b]$, given by

   $$Au := u',$$

   where u' is the derivative of u. Show that A is closed.

   (c) Let $X=Y=L^p(I)$, where $I\subseteq\mathbb{R}$ is an open interval and $p\in[1,\infty]$. Consider the linear operator $A: D(A)\to Y$ with domain $D(A)=W^{1,p}(I)$ given by

   $$Au := u',$$

   where u' is the weak derivative of u. Show that A is closed.

   Solution: Was given in the assignment class.

3. Show that the weak solution $u \in H_0^1(0,1) \cap H^2(0,1)$ of the boundary value problem

$$\begin{cases} -u'' = f, \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

for a fixed $f \in L^2(0,1)$, satisfies

$$J(u) = \min \{ J(v) : \ v \in H_0^1(0,1) \},$$

where $J : H_0^1(0,1) \to \mathbb{R}$ is the 'energy functional' given by

$$J(v) = \frac{1}{2} \int_0^1 (v'(x))^2 \, dx - \int_0^1 f(x) v(x) \, dx.$$

Hint: Compute $J(u+h)$ for arbitrary $h \in H_0^1(0,1)$.

Solution: Any $v \in H_0^1(0,1)$ can be written as $v = u + h$, where u is the weak solution and $h = v - u$ is in $H_0^1(0,1)$ since both u and v are. Then

$$\begin{aligned} J(u+h) &= \frac{1}{2} \int_0^1 (u'(x) + h'(x))^2 \, dx - \int_0^1 f(x)(u(x) + h(x)) \, dx \\ &= J(u) + \int_0^1 u'(x) h'(x) \, dx + \int_0^1 (h'(x))^2 \, dx - \int_0^1 f(x) h(x) \, dx \\ &\overset{(*)}{=} J(u) - \int_0^1 u''(x) h(x) \, dx + \int_0^1 (h'(x))^2 \, dx - \int_0^1 f(x) h(x) \, dx \\ &= J(u) + \int_0^1 (h'(x))^2 \, dx \geqslant J(u). \end{aligned}$$

This shows that the weak solution u minimises J. (correcte berekening: $1\frac{1}{2}$ pt)

The integration by parts step in $(*)$ is justified as follows. Since the second weak derivative u'' equals $-f$ we have

$$\int_0^1 u'(x) \phi'(x) \, dx = - \int_0^1 u''(x) \phi(x) \, dx \quad \forall \phi \in C_c^1(0,1). \tag{2}$$

Now given an $h \in H_0^1(0,1)$, using the density of $C_c^1(0,1)$ in $H_0^1(0,1)$ (zien dat een dichtheidsargument nodig is: $\frac{1}{2}$ pt) we can find $\phi_n \in C_c^1(0,1)$ such that $\phi_n \to h$ in the norm of $H_0^1(0,1)$, that is, $\phi_n \to h$ in $L^2(0,1)$ and $\phi_n' \to h'$ in $L^2(0,1)$. But then, by Cauchy–Schwarz (note that u' and u'' are in $L^2(0,1)$),

$$\int_0^1 u'(x) \phi_n'(x) \, dx \to \int_0^1 u'(x) h'(x) \, dx$$

and

$$\int_0^1 u''(x) \phi_n(x) \, dx \to \int_0^1 u''(x) h(x) \, dx.$$

Therefore, applying (2) to ϕ_n and taking the limit $n \to \infty$ on both sides, we arrive at (correct dichtheidsargument: 1 pt)

$$\int_0^1 u'(x)h'(x)\,dx = -\int_0^1 u''(x)h(x)\,dx.$$

`-- The end --`