TU Delft Mekelweg 4
Faculteit EWI, DIAM 2628 CD Delft

Assignment exam Applied Functional Analysis
October 23, 2015, 15.45 - 17.30

Solutions should be given in full detail

Grading: $\frac{1}{5}((4+6)+10+(6+4)+((2+4+4+6)+4))$

All vector spaces are over the scalar field $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$.

1. Let X be a Banach space.

   (a) Show that every absolutely convergent series in X is convergent.
   Solution: As in the assignment class (4 pt).

   (b) Suppose that Y is another Banach space and let $T : X \to Y$ be a bounded linear operator such that

$$\sum_{n=0}^{\infty} \|T^n x\| < \infty \quad \forall x \in X.$$

   Prove that

$$Sx := \sum_{n=0}^{\infty} T^n x, \quad x \in X,$$

   defines a bounded linear operator $S : X \to Y$.
   Hint: Apply the uniform boundedness theorem.
   Solution: The sum defining Sx converges for every $x \in X$ by (a). It is clear that $x \mapsto Sx$ is linear (1 pt). For all $x \in X$ the sequence $(\sum_{k=0}^{n} T^k x)_{n=0}^{\infty}$ is bounded, so the uniform boundedness theorem implies that the operators $S_n := \sum_{k=0}^{n} T^k$ are uniformly bounded, say $\|S_n\| \leqslant M$. (3 pt) Then $\|Sx\| = \lim_{n\to\infty} \|S_n x\| \leqslant M\|x\|$, proving that S is bounded with norm $\|S\| \leqslant M$ (2 pt).

2. Define the linear operator $T : L^1(\mathbb{R}) \to C_{\mathrm{b}}(\mathbb{R})$ (the space of all bounded continuous functions on $\mathbb{R}$) by

$$(Tf)(s) := \int_{-\infty}^{s} f(t)\,dt, \quad s \in \mathbb{R}.$$

   Show that T is well-defined (i.e., takes values in $C_{\mathrm{b}}(\mathbb{R})$) and bounded, and compute its operator norm.

Solution: Tf is sequentially continuous by the dominated convergence theorem, and therefore continuous, for every $f \in L^1(\mathbb{R})$ (3 pt). Also,

$$|Tf(s)| \leqslant \int_{-\infty}^{s} |f(t)|\,dt \leqslant \int_{\mathbb{R}} |f(t)|\,dt = \|f\|_1,$$

so after taking the supremum over all $s \in \mathbb{R}$ we attive at $\|T\| \leqslant 1$ (4 pt). So see that $\|T\| = 1$ take $f = 1_{[a,b]}$. Then $\|f\|_1 = b - a$ and $Tf(s) = 0$ for $s < a$, $= s - a$ for $s \in [a, b]$, and $= b - a$ for $s > b$, so $\|Tf\|_\infty = b - a$ (3 pt).

3. Let X be a Banach space and define $X^* := \mathscr{L}(X, \mathbb{K})$. This is a Banach space with respect to the norm

$$\|x^*\| = \sup_{\|x\| \leqslant 1} |\langle x, x^* \rangle|,$$

where we use the notation $\langle x, x^* \rangle := x^*(x)$. Suppose (Ω, μ) is a measure space and let $f : \Omega \to X$ be a function which has the property that $\omega \mapsto \langle f(\omega), x^* \rangle$ is integrable for every $x^* \in X^*$.

   (a) Show that the linear map $T : X^* \to L^1(\Omega, \mu)$ defined by $x^* \mapsto \langle f(\cdot), x^* \rangle$ is closed.
   Solution: As in the assignment class (6 pt).

   (b) Deduce that there exists a finite constant $C \geqslant 0$ such that

$$\|\langle f(\cdot), x^* \rangle\|_{L^1(\Omega,\mu)} \leqslant C\|x^*\|$$

   for all $x^* \in X^*$.
   Solution: As in the assignment class (4 pt).

4. Let $1 \leqslant p < \infty$ and let $t \in \mathbb{R}$ be fixed. Consider the shift operator $T_t \in \mathscr{L}(L^p(\mathbb{R}))$ given by

$$T_t f(s) = f(s - t), \quad s \in \mathbb{R}.$$

   (a) Prove that $\sigma(T_t) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ by completing the following steps:
      (i) If $\lambda = 0$, then $\lambda \notin \sigma(T_t)$.
      (ii) If $|\lambda| > 1$, then $\lambda \notin \sigma(T_t)$.
      (iii) If $0 < |\lambda| < 1$, then $\lambda \notin \sigma(T_t)$.
      Hint: First show that $\lambda^{-1} - T_t^{-1}$ is invertible and find its inverse.
      (iv) If $|\lambda| = 1$, then $\lambda \in \sigma(T_t)$.
      Hint: Show that if $(\lambda - T_t)f = g$, then for all $n = 0, 1, 2, \ldots$

$$f(s) = \frac{1}{\lambda^{n+1}} f(s - (n+1)t) + \sum_{k=0}^{n} \frac{1}{\lambda^{k+1}} g(s - kt).$$

      Show that the first term on the right-hand side tends to 0 in $L^p(\mathbb{R})$ as $n \to \infty$. Explain why the assumption $\lambda \in \varrho(T)$ now leads to a contradiction.

(b) Does T_t have eigenvalues?

Solution: (a) - in (i) the inverse is T_{-t} (2 pt), (ii) follows from the Neumann series (2 pt), noting that $\|T_t\| \leqslant 1$ (2 pt), and (iii) follows by noting that $-\lambda^{-1}T_t^{-1}(\lambda^{-1} - T_t^{-1})^{-1}$ is a two-sided inverse of $\lambda - T_t$ (this identity can be found by using the Neumann series or noting that for scalars we have $\frac{1}{\lambda} - \frac{1}{a} = -\frac{\lambda a}{\lambda - a}$) (4 pt).

For (iv) let $|\lambda| = 1$. Consider the equation $(\lambda - T)f = g$ with $g \in L^p(\mathbb{R})$ given. This can be rewritten as

$$f(s) = \frac{1}{\lambda}(g(s) + f(s-t)), \quad s \in \mathbb{R}.$$

Also, $\lambda f(s-t) - f(s-2t) = g(s-t)$, $s \in \mathbb{R}$, so

$$f(s) = \frac{1}{\lambda}(g(s) + f(s-t)) = \frac{1}{\lambda}g(s) + \frac{1}{\lambda^2}(g(s-t) + f(s-2t)).$$

Continuing in this way we obtain

$$f(s) = \frac{1}{\lambda}g(s) + \frac{1}{\lambda^2}g(s-t) + \cdots + \frac{1}{\lambda^{n+1}}(g(s-nt) + f(s-(n+1)t)).$$

(2 pt) As $n \to \infty$, the last term satisfies

$$\left\| \frac{1}{\lambda^{n+1}} f(\cdot - (n+1)t)) \right\|_p = \left\| f(\cdot - (n+1)t)) \right\|_p \to 0$$

by dominated convergence, using that $|\lambda| = 1$ in the first step (2 pt). Now if $(\lambda - T_t)f = g$ has a solution $f \in L^p(\mathbb{R})$, it follows that in $L^p(\mathbb{R})$ it is given by the convergent series

$$f = \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} g(\cdot - nt).$$

But the terms of a convergent series must converge to 0; here we have $\|\frac{1}{\lambda^{n+1}} g(\cdot - nt)\|_p = \|g\|_p \not\to 0$ (2 pt).

(b) - No. If f is a non-zero eigenfunction for some $|\lambda| = 1$, then the definition of T_t shows that its modulus $|f|$ is t-periodic (2 pt). But then f cannot be in $L^p(\mathbb{R}^d)$ (2 pt).