TU Delft Mekelweg 4
Faculteit EWI, DIAM 2628 CD Delft

# Exam Applied Functional Analysis
## January 24, 2014, 14.00 - 17.00

All answers should be carefully motivated

Grading: $(1+1)+(1+\frac{1}{2})+(1+1)+(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2})+(1+\frac{1}{2})+(1 \text{ free})$

*All vector spaces in this exam are real.*

1. Let $E = \{f \in C[0,1] : f(1) = 0\}$ endowed with the supremum norm $\|\cdot\|_\infty$. Consider the functional $\phi : E \to \mathbb{R}$ defined by
$$\phi(f) = \int_0^1 f(t)\,dt \quad (f \in E).$$
   (a) Show that $\phi$ is bounded and determine its norm.
   (b) Show that for all non-zero $f \in E$ we have $|\phi(f)| < \|f\|_\infty$.

2. Let $(\Omega, d)$ be a metric space, let $f : \Omega \to \mathbb{R}$ be continuous, and consider a subset $A \subseteq \Omega$.
   (a) Show that $f(\overline{A}) \subseteq \overline{f(A)}$.
   (b) Show that if $A$ is compact, then $f(A)$ is compact.

3. Let $(h_n)_{n=1}^\infty$ be an orthonormal system in a Hilbert space $H$.
   (a) Prove that for all $h \in H$ we have $\sum_{n=1}^\infty |\langle h, f_n\rangle|^2 \leqslant \|h\|^2$.
   *Hint:* Write $h = (h - P_N h) + P_N h$, where $P_N$ is the orthogonal projection onto the linear span of $(h_n)_{n=1}^N$, and prove first that $\|P_N h\| \leqslant \|h\|$.
   (b) Prove that the following three assertions are equivalent:
      (i) the linear span of $(h_n)_{n=1}^\infty$ is dense in $H$;
      (ii) for all $h \in H$ the sum $\sum_{n=1}^\infty \langle h, h_n\rangle h_n$ converges to $h$ in $H$;
      (iii) for all $h \in H$ we have $\sum_{n=1}^\infty |\langle h, h_n\rangle|^2 = \|h\|^2$.
   *Hint:* One could proceed by proving (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(i). For (i)$\Rightarrow$(ii) show first that $P_N h \to h$ for each $h$ in the (dense) linear span of $(h_n)_{n=1}^\infty$; for (iii)$\Rightarrow$(i) consider a vector $g$ orthogonal to each $h_n$.

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Fundamental Thm of Calculus
Corollary 10.7

4. Consider the operator of indefinite integration $J : L^2(a,b) \to C[a,b]$,

$$Jf(t) := \int_a^t f(s)\,ds \quad (f \in L^2(a,b),\ t \in [a,b]).$$

   (a) Show that $J$ indeed takes values in $C[a,b]$ and is bounded as an operator from $L^2(a,b)$ to $C[a,b]$.

   (b) Show that for all $f \in L^2(a,b)$ the function $Jf$ is in $H^1(a,b)$, with weak derivative $(Jf)' = f$.

   (c) Show that a function $g \in L^2(a,b)$ belongs to $H^1(a,b)$ if and only if it is of the form $g = Jf + c\mathbf{1}$ for some $f \in L^2(a,b)$ and $c \in \mathbb{R}$; here $\mathbf{1}$ is the constant-one function.

   (d) Use (c) to prove the following version of the Poincaré inequality: there exists a finite constant $c \geqslant 0$ for all $g \in H^1(0,1)$ we have

$$\|g - \langle g, \mathbf{1}\rangle_{L^2(0,1)}\mathbf{1}\|_{L^2(0,1)} \leqslant c\|g'\|_{L^2(0,1)}.$$

5. Let $X$ and $Y$ be Banach spaces. A linear operator $T : X \to Y$ is called *compact* if the image under $T$ of every bounded sequence in $X$ has a convergent subsequence in $Y$.

   (a) Show that every compact operator is bounded.

   (b) Show that if $T : X \to Y$ is compact and $U : X \to X$ and $S : Y \to Y$ are bounded linear operators, on $X$ and $Y$, then $S \circ T \circ U : X \to Y$ is compact.