2628 CD Delft

## Exam Applied Functional Analysis January 24, 2014, 14.00 - 17.00

All answers should be carefully motivated

Grading: 
$$(1+1)+(1+\frac{1}{2})+(1+1)+(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2})+(1+\frac{1}{2})+(1$$
 free)

All vector spaces in this exam are real.

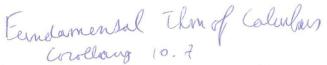
1. Let  $E = \{f \in C[0,1]: f(1) = 0\}$  endowed with the supremum norm  $\|\cdot\|_{\infty}$ . Consider the functional  $\phi: E \to \mathbb{R}$  defined by

$$\phi(f) = \int_0^1 f(t) dt \quad (f \in E).$$

- (a) Show that  $\phi$  is bounded and determine its norm.
- (b) Show that for all non-zero  $f \in E$  we have  $|\phi(f)| < ||f||_{\infty}$ .
- 2. Let  $(\Omega, d)$  be a metric space, let  $f: \Omega \to \mathbb{R}$  be continuous, and consider a subset  $A \subseteq \Omega$ .
  - (a) Show that  $f(\overline{A}) \subset \overline{f(A)}$ .
  - (b) Show that if A is compact, then f(A) is compact.
- 3. Let  $(h_n)_{n=1}^{\infty}$  be an orthonormal system in a Hilbert space H.
  - (a) Prove that for all  $h \in H$  we have  $\sum_{n=1}^{\infty} |\langle h, f_n \rangle|^2 \leq ||h||^2$ . Hint: Write  $h = (h - P_N h) + P_N h$ , where  $P_N$  is the orthogonal projection onto the linear span of  $(h_n)_{n=1}^N$ , and prove first that  $||P_N h|| \leqslant ||h||.$
  - (b) Prove that the following three assertions are equivalent:
    - (i) the linear span of  $(h_n)_{n=1}^{\infty}$  is dense in H;
    - (ii) for all  $h \in H$  the sum  $\sum_{n=1}^{\infty} \langle h, h_n \rangle h_n$  converges to h in H; (iii) for all  $h \in H$  we have  $\sum_{n=1}^{\infty} |\langle h, h_n \rangle|^2 = ||h||^2$ .

*Hint*: One could proceed by proving  $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$ . For  $(i) \Rightarrow (ii)$ show first that  $P_N h \to h$  for each h in the (dense) linear span of  $(h_n)_{n=1}^{\infty}$ ; for (iii) $\Rightarrow$ (i) consider a vector g orthogonal to each  $h_n$ .

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4. Consider the operator of indefinite integration  $J: L^2(a,b) \to C[a,b]$ ,

$$Jf(t) := \int_a^t f(s) \, ds \quad (f \in L^2(a,b), \ t \in [a,b]).$$

- (a) Show that J indeed takes values in C[a,b] and is bounded as an operator from  $L^2(a,b)$  to C[a,b].
- (b) Show that for all  $f \in L^2(a,b)$  the function Jf is in  $H^1(a,b)$ , with weak derivative (Jf)' = f.
- (c) Show that a function  $g \in L^2(a,b)$  belongs to  $H^1(a,b)$  if and only if it is of the form g = Jf + c1 for some  $f \in L^2(a,b)$  and  $c \in \mathbb{R}$ ; here 1 is the constant-one function.
- (d) Use (c) to prove the following version of the Poincaré inequality: there exists a finite constant  $c \ge 0$  for all  $g \in H^1(0,1)$  we have

$$||g - \langle g, \mathbf{1} \rangle_{L^2(0,1)} \mathbf{1}||_{L^2(0,1)} \le c ||g'||_{L^2(0,1)}.$$

- 5. Let X and Y be Banach spaces. A linear operator  $T: X \to Y$  is called compact if the image under T of every bounded sequence in X has a convergent subsequence in Y.
  - (a) Show that every compact operator is bounded.
  - (b) Show that if  $T:X\to Y$  is compact and  $U:X\to X$  and  $S:Y\to Y$  are bounded linear operators, on X and Y, then  $S\circ T\circ U:X\to Y$  is compact.