Dear all,

The solutions given in these notes differ in two places from the ones given last Monday when I worked out the example exam.

Firstly, the answer given in Assignment 2b, the Fourier transform of $x(2n+1)$, was not correct. I took this example from the book (Problem 4.22a, p. 298), including the answer given in the solution manual. While carefully writing out the solutions for these notes, I realized that there was a mistake in the solutions given by Proakis. Actually, the right answer was already given during the instructions on October 6 where a similar problem (Problem 4.17e, p. 297) was discussed. At that time I didn't have the solution manual, which nicely shows that you can better think for yourself, rather than copying results from others ☺

The second difference is in Assignment 3, in particular 3a. Although the results are right, the way I derived the result is not correct. To avoid these problems (which I will explain below), I changed the question as follows:

Consider the continuous-time signal

$$x_a(t) = \begin{cases} e^{-\alpha t} e^{-j2\pi f_0 t}, & t \geq 0 \\ 0, & t < 0 \end{cases}, \quad \alpha > 0.$$

a) Show that

$$X_a(f) = \frac{1}{j2\pi(f+f_0)+\alpha}.$$

Note that for $\alpha \to 0$, we are back at the problem as it was originally posted.

The problem we have when $\alpha = 0$ is that that signal has infinite energy so that it is tricky to compute the Fourier transform (the function is not integrable). Indeed, the energy of $x_a(t)$ as given above is

$$\int_{-\infty}^{\infty} |x_a(t)|^2 dt = \int_0^{\infty} e^{-2\alpha t} dt = \frac{-1}{2\alpha} e^{-2\alpha t} \Big|_0^{\infty} = \frac{1}{2\alpha},$$

which is finite for $\alpha > 0$, but becomes infinte for $\alpha = 0$.

Let us have a look at the solution of Assignment 3a in these notes. We have

$$\begin{aligned}\int_0^\infty e^{-(j2\pi(f+f_0)+\alpha)t}dt &= \frac{-1}{j2\pi(f+f_0)+\alpha}\; e^{-(j2\pi(f+f_0)+\alpha)t}\Big|_0^\infty \\ &= 0 - \frac{-1}{j2\pi(f+f_0)+\alpha} \\ &= \frac{1}{j2\pi(f+f_0)+\alpha}.\end{aligned}$$

This derivation is *only* valid for $\alpha > 0$ *not* for $\alpha = 0$. So,

$$\frac{-1}{j2\pi(f+f_0)}\; e^{-j2\pi(f+f_0)t}\Big|_0^\infty \neq 0 - \frac{-1}{j2\pi(f+f_0)},$$

as I wrote down last week. The reason for this is that

$$\lim_{t\to\infty} e^{-jt} \neq 0,$$

since the function e^{-jt} is 2π-periodic. This in contrast to

$$\lim_{t\to\infty} e^{-t} = 0.$$

Now coming back to the solution of Assignment 3a, we conclude that

$$\begin{aligned}\frac{-1}{j2\pi(f+f_0)+\alpha}\; e^{-(j2\pi(f+f_0)+\alpha)t}\Big|_0^\infty &= \frac{-1}{j2\pi(f+f_0)-\alpha}\; e^{-j2\pi(f+f_0)t}e^{-\alpha t}\Big|_0^\infty \\ &= 0 - \frac{-1}{j2\pi(f+f_0)-\alpha},\end{aligned}$$

if and only if $\alpha > 0$ since in that case the term $e^{-\alpha t} \to 0$ as $t \to \infty$ (note that $|e^{-j2\pi(f+f_0)t}| = 1$).

In order to correctly prove the result for $\alpha = 0$, we have to use a limiting argument. That is,

$$x_a(t) = \begin{cases} e^{-j2\pi f_0 t}, & t \geq 0 \\ 0, & t < 0 \end{cases} = \lim_{\alpha\downarrow 0} \begin{cases} e^{-\alpha t}e^{-j2\pi f_0 t}, & t \geq 0 \\ 0, & t < 0 \end{cases},$$

so that

$$X_a(f) = \lim_{\alpha\downarrow 0} \frac{1}{j2\pi(f+f_0)+\alpha} = \frac{1}{j2\pi(f+f_0)}, \quad f \in \mathbb{R} \setminus \{-f_0\}.$$

Some additonal remarks. I do not expect you to derive Fourier transforms using limiting arguments. When writing down the original assignment in the weekend I simply overlooked the fact that the signal was of infinite energy. The signals you can expect at the exam will be of finite energy.

Richard

Assignment 1:

a)

$$H(z) = \frac{-z}{z^2 - 2z + \frac{3}{4}} = \frac{-z}{(z-\frac{1}{2})(z-\frac{3}{2})}.$$

Hence, a zero at $z = 0$ and two poles at $z = \frac{1}{2}$ and $z = \frac{3}{2}$.

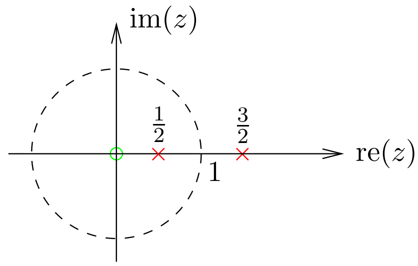

Figure 1: Pole-zero map of $H(z)$.

b) ROC $|z| > \frac{3}{2}$. Hence, we have a causal solution and

$$h(n) = \frac{1}{2\pi j}\oint_C H(z)z^{n-1}dz = \frac{1}{2\pi j}\oint_C \frac{-z^n}{(z-\frac{1}{2})(z-\frac{3}{2})}dz,$$

where C is a counterclockwise closed contour in the region of convergence $|z| > \frac{3}{2}$. For $n \geq 0$, we have two poles inside C, so that

$$\begin{aligned} h(n) &= \operatorname*{Res}_{z=\frac{1}{2}} H(z)z^{n-1} + \operatorname*{Res}_{z=\frac{3}{2}} H(z)z^{n-1} \\ &= \lim_{z\to\frac{1}{2}}(z-\frac{1}{2})H(z)z^{n-1} + \lim_{z\to\frac{3}{2}}(z-\frac{3}{2})H(z)z^{n-1} \\ &= \lim_{z\to\frac{1}{2}}\frac{-z^n}{z-\frac{3}{2}} + \lim_{z\to\frac{3}{2}}\frac{-z^n}{z-\frac{1}{2}} = \left(\frac{1}{2}\right)^n - \left(\frac{3}{2}\right)^n. \end{aligned}$$

c) ROC $|z| < \frac{1}{2}$. Hence, we have an anti-causal solution.

$$H(z) = \frac{-z}{(z-\frac{1}{2})(z-\frac{3}{2})} = \frac{-z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{3}{2}z^{-1})},$$

so that, with $p = z^{-1}$,

$$H(p) = \frac{-p}{(1-\frac{1}{2}p)(1-\frac{3}{2}p)} = \frac{-\frac{4}{3}p}{(p-2)(p-\frac{2}{3})}.$$

Hence, $H(p)$ has poles at $p = \frac{2}{3}$ and $p = 2$, which both lie inside the region of convergence, which is $|p| > 2$ ($|z| < \frac{1}{2}$ implies $|p| > 2$).

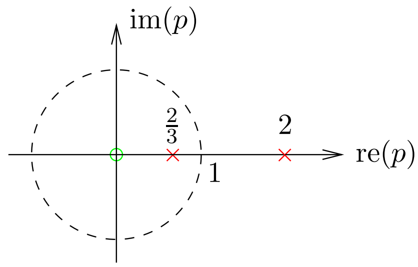

Figure 2: Pole-zero map

Since we have an anti-causal solution, we use

$$h(n) = \frac{1}{2\pi j}\oint_{C'} H(p)p^{-n-1}dp = \frac{1}{2\pi j}\oint_{C'} \frac{-\frac{4}{3}p^{-n}}{(p-2)(p-\frac{2}{3})}dp,$$

where C' is a counterclockwise closed contour in the region of convergence $|p| > 2$. For $n \leq 0$, we have two poles inside C' so that

$$\begin{aligned} h(n) &= \operatorname*{Res}_{p=\frac{2}{3}} H(p)p^{-n-1} + \operatorname*{Res}_{p=2} H(p)p^{-n-1} \\ &= \lim_{p\to\frac{2}{3}} \frac{-\frac{4}{3}p^{-n}}{p-2} + \lim_{p\to 2} \frac{-\frac{4}{3}p^{-n}}{p-\frac{2}{3}} = \left(\frac{3}{2}\right)^n - \left(\frac{1}{2}\right)^n. \end{aligned}$$

d) ROC $\frac{1}{2} < |z| < \frac{3}{2}$. In this case we have a causal and anti-causal contribution to the total solution. We have

$$h(n) = \underbrace{\frac{1}{2\pi j}\oint_C \frac{-z^n}{(z-\frac{1}{2})(z-\frac{3}{2})}dz}_{\text{1 pole inside } C \text{ for } n \geq 0} = \underbrace{\frac{1}{2\pi j}\oint_{C'} \frac{-\frac{4}{3}p^{-n}}{(p-2)(p-\frac{2}{3})}dp}_{\text{1 pole inside } C' \text{ for } n \leq 0},$$

where C is taken in $\frac{1}{2} < |z| < \frac{3}{2}$ and C' in $\frac{2}{3} < |p| < 2$. Hence,

$$n \geq 0: \quad h(n) = \operatorname*{Res}_{p=\frac{1}{2}} H(z) z^{n-1} \overset{\text{(see b)}}{=} \left(\frac{1}{2}\right)^n,$$

$$n \leq 0: \quad h(n) = \operatorname*{Res}_{p=\frac{2}{3}} H(p) p^{-n-1} \overset{\text{(see c)}}{=} \left(\frac{3}{2}\right)^n.$$

Note that we have a stable solution in case the region of convergence is $\frac{1}{2} < |z| < \frac{3}{2}$, since this is the only region containing the unit circle.

Alternative solution: Partial-fraction expansion

$$\frac{H(z)}{z} = \frac{-1}{(z - \frac{1}{2})(z - \frac{3}{2})} = \frac{A}{z - \frac{1}{2}} + \frac{B}{z - \frac{3}{2}}.$$

The constants A and B are found by

$$A = \lim_{z \to \frac{1}{2}} (z - \frac{1}{2}) \frac{H(z)}{z} = \lim_{z \to \frac{1}{2}} \frac{-1}{z - \frac{3}{2}} = 1,$$

$$B = \lim_{z \to \frac{3}{2}} (z - \frac{3}{2}) \frac{H(z)}{z} = \lim_{z \to \frac{3}{2}} \frac{-1}{z - \frac{1}{2}} = -1,$$

and we conclude that

$$H(z) = \frac{z}{z - \frac{1}{2}} - \frac{z}{z - \frac{3}{2}}.$$

b) ROC $|z| > \frac{3}{2}$ and thus $|z| > \frac{1}{2}$. By table lookup (see Table 3.3, p. 170) we find that

$$a^n u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - a z^{-1}} = \frac{z}{z - a}, \quad |z| > a,$$

so that

$$h(n) = \left(\left(\frac{1}{2}\right)^n - \left(\frac{3}{2}\right)^n\right) u(n).$$

c) ROC $|z| < \frac{1}{2}$ and thus $|z| < \frac{3}{2}$. By table lookup we find that

$$-a^n u(-n-1) \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \quad |z| < a,$$

so that

$$h(n) = \left(\left(\frac{3}{2}\right)^n - \left(\frac{1}{2}\right)^n \right) u(-n-1).$$

d) ROC $\frac{1}{2} < |z| < \frac{3}{2}$. By table lookup we find

$$h(n) = \left(\frac{1}{2}\right)^n u(n) + \left(\frac{3}{2}\right)^n u(-n-1).$$

Assignment 2:

a) We know that

$$\frac{1}{1-az^{-1}} \overset{\mathcal{Z}}{\longleftrightarrow} a^n u(n), \quad |z| > a,$$

and $X(\omega) = X(z)|_{z=e^{j\omega}}$, so that we conclude $x(n) = a^n u(n)$.

b) We define $y(n) = x(2n)$ so that $y(n+\frac{1}{2}) = x(2n+1)$. From Table 4.5, p. 290, we conclude that

$$y(n+\frac{1}{2}) \overset{\mathcal{F}}{\longleftrightarrow} e^{j\frac{\omega}{2}} Y(\omega),$$

so that we are left with finding an expression for $Y(\omega)$.

$$\begin{aligned}
Y(\omega) &= \sum_{n=-\infty}^{\infty} y(n) e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x(2n) e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} x(2n) e^{-j\frac{\omega}{2}2n} = \sum_{m \text{ even}} x(m) e^{-j\frac{\omega}{2}m} \\
&= \sum_{m=-\infty}^{\infty} \frac{1}{2}(1+(-1)^m) x(m) e^{-j\frac{\omega}{2}m} \\
&= \frac{1}{2}\sum_{m=-\infty}^{\infty} x(m) e^{-j\frac{\omega}{2}m} + \frac{1}{2}\sum_{m=-\infty}^{\infty} (-1)^m x(m) e^{-j\frac{\omega}{2}m} \\
&= \frac{1}{2}\sum_{m=-\infty}^{\infty} x(m) e^{-j\frac{\omega}{2}m} + \frac{1}{2}\sum_{m=-\infty}^{\infty} x(m) e^{-j(\frac{\omega}{2}-\pi)m} \\
&= \frac{1}{2} X\left(\frac{\omega}{2}\right) + \frac{1}{2} X\left(\frac{\omega}{2}-\pi\right).
\end{aligned}$$

Hence,

$$x(2n+1) \overset{\mathcal{F}}{\longleftrightarrow} \frac{e^{j\frac{\omega}{2}}}{2}\left(X\left(\frac{\omega}{2}\right) + X\left(\frac{\omega}{2}-\pi\right)\right).$$

c) From Table 4.5 we conclude that

$$(x * x)(n) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)X(\omega) = X^2(\omega).$$

d) Again from Table 4.5 we conclude that

$$x(n)\cos(\frac{\pi}{3}n) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}X(\omega-\frac{\pi}{3})+\frac{1}{2}X(\omega+\frac{\pi}{3}).$$

Assignment 3:

a) We have

$$\begin{aligned}
X_a(f) &= \int_{-\infty}^{\infty} x_a(t) e^{-j2\pi f t} dt = \int_0^{\infty} e^{-(j2\pi(f+f_0)+\alpha)t} dt \\
&= \frac{-1}{j2\pi(f+f_0)+\alpha} \, e^{-(j2\pi(f+f_0)+\alpha)t} \Big|_0^{\infty} \\
&= 0 - \frac{-1}{j2\pi(f+f_0)+\alpha} \\
&= \frac{1}{j2\pi(f+f_0)+\alpha}.
\end{aligned}$$

b)

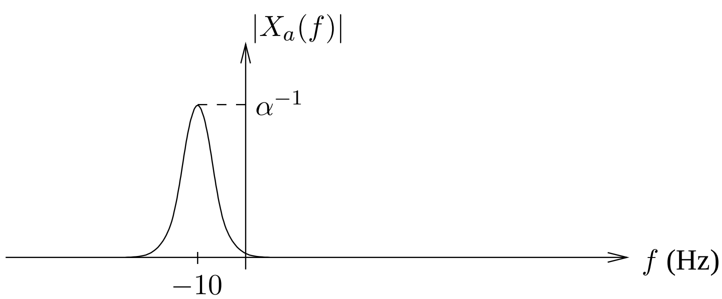

Figure 3: Magnitude spectrum of $X_a(f)$.

c) Due to the spectral overlap, aliasing errors occur. However, if the sampling frequency increases, this error will become smaller. In the example at hand, a sampling frequency of $f_s = 40$ Hz will already give reasonable results.

d) Since $X_a(f)$ is *not* bandlimited, we cannot recover $x_a(t)$ out of its samples $x(n)$.

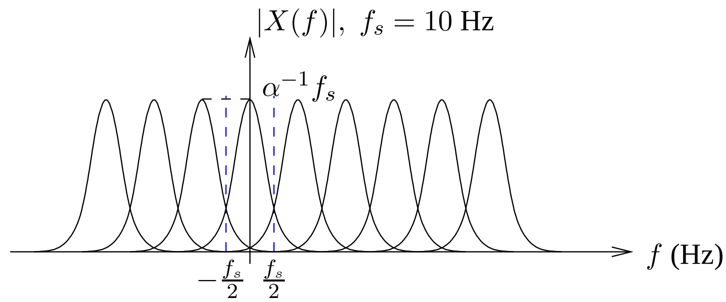

Figure 4: Magnitude spectrum of $X(f)$ when $f_s = 10$ Hz.

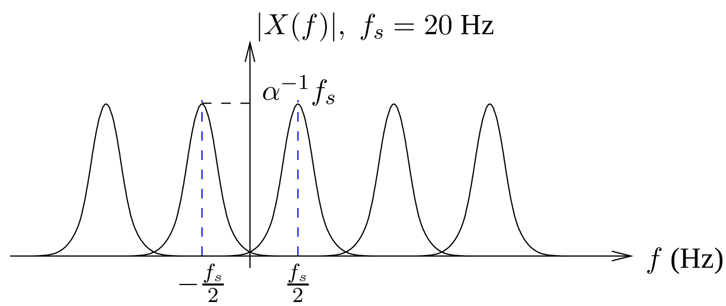

Figure 5: Magnitude spectrum of $X(f)$ when $f_s = 20$ Hz.

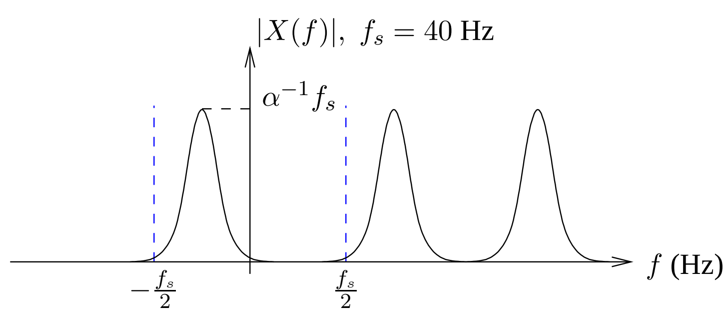

Figure 6: Magnitude spectrum of $X(f)$ when $f_s = 40$ Hz.

Assignment 4:

a)

$$H(z) = \frac{z^2 - 1}{z^2 + \frac{1}{4}} = \frac{1 - z^{-2}}{1 + \frac{1}{4}z^{-2}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}.$$

Hence, $b_0 = 1, b_1 = 0, b_2 = -1, a_1 = 0$ and $a_2 = \frac{1}{4}$.

b) By inspection of the stability triangle (see Figure 3.5.1, p. 202), we conclude that the system is stable ($a_1 < 1 + a_2$ and $a_2 < 1$). Moreover, since $a_2 > \frac{a_1^2}{4}$, we conclude that the system has two complex-conjugated poles.

c)

$$H(z) = \frac{(z+1)(z-1)}{(z + \frac{1}{2}j)(z - \frac{1}{2}j)}.$$

We have two zeros at $z = \pm 1$ and two poles at $z = \pm\frac{1}{2}j$.

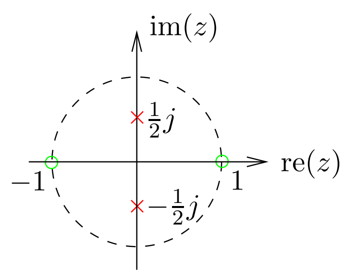

Figure 7: Pole-zero map of $H(z)$.

d) Since the system has two zeros at $z \pm 1$, the magnitude response will be zero at $\omega = 0$ and $\omega = \pi$. It reaches a maximum at frequencies $\omega = \frac{\pi}{2}$ and $\omega = \frac{3\pi}{2}$ (closest to the poles) of

$$\frac{|e^{j\frac{\pi}{2}} + 1| \cdot |e^{j\frac{\pi}{2}} - 1|}{|e^{j\frac{\pi}{2}} + \frac{1}{2}j| \cdot |e^{j\frac{\pi}{2}} - \frac{1}{2}j|} = \frac{\sqrt{2}\sqrt{2}}{\frac{3}{2}\frac{1}{2}} = \frac{8}{3}.$$

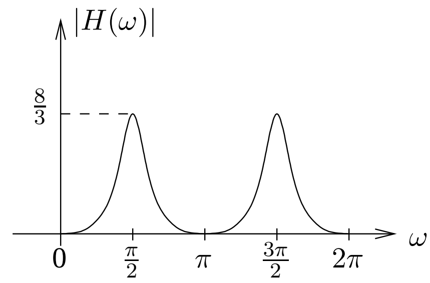

Figure 8: Magnitude spectrum of $H(\omega)$.

The phase response is given by

$$\angle H(\omega) = \angle(e^{j\omega}+1) + \angle(e^{j\omega}-1) - \angle(e^{j\omega}+\frac{1}{2}j) - \angle(e^{j\omega}-\frac{1}{2}j).$$

We have at $\omega = 0^+$ that

$$\angle H(0^+) = 0 + \frac{\pi}{2} - \angle(1+\frac{1}{2}j) - \angle(1-\frac{1}{2}j) = \frac{\pi}{2},$$

and at $\omega = 0^-$

$$\angle H(0^-) = 0 - \frac{\pi}{2} - \angle(1+\frac{1}{2}j) - \angle(1-\frac{1}{2}j) = -\frac{\pi}{2}.$$

At $\omega = \frac{\pi}{2}$ we find

$$\begin{aligned}\angle H(\frac{\pi}{2}) &= \angle(j+1) + \angle(j-1) - \angle(j+\frac{1}{2}j) - \angle(j-\frac{1}{2}j)\\ &= \frac{\pi}{4} + \frac{3\pi}{4} - \frac{\pi}{2} - \frac{\pi}{2} = 0.\end{aligned}$$

Similarly, we have $\angle H(\frac{3\pi}{2}) = 0$.

e) The steady-state response is given by

$$y_{\text{ss}}(n) = H(\frac{\pi}{2})e^{j\frac{\pi}{2}n} + H(\pi)e^{i\pi n} = \frac{8}{3}e^{j\frac{\pi}{2}n},$$

since $|H(\frac{\pi}{2})| = \frac{8}{3},\ \angle H(\frac{\pi}{2}) = 0$ and $H(\pi) = 0$.

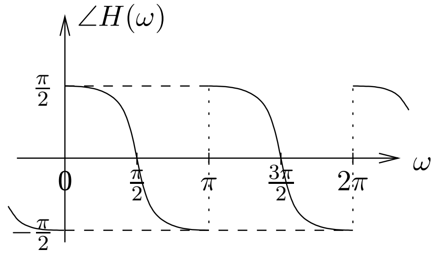

Figure 9: Phase spectrum of $H(\omega)$.

f) Let $x(n) = x_1(n) + x_2(n)$, where $x_1(n) = e^{j\frac{\pi}{2}n}$ and $x_2(n) = e^{i\pi n}$. We then have that

$$Y(z) = H(z)X(z) = H(z)X_1(z) + H(z)X_2(z),$$

with

$$X_1(z) = \frac{z}{z - e^{j\frac{\pi}{2}}} = \frac{z}{z-j},$$

$$X_2(z) = \frac{z}{z - e^{j\pi}} = \frac{z}{z+1}.$$

Therefore,

$$Y_1(z) = H(z)X_1(z) = \frac{z(z^2-1)}{(z^2+\frac{1}{4})(z-j)} = \frac{z(z^2-1)}{(z+\frac{1}{2}j)(z-\frac{1}{2}j)(z-j)}.$$

The signal $y_1(n)$ can be found using contour integration or partial-fraction expansion.

Partial-fraction expansion:

$$\frac{Y_1(z)}{z} = \frac{z^2-1}{(z+\frac{1}{2}j)(z-\frac{1}{2}j)(z-j)} = \frac{A}{z+\frac{1}{2}j} + \frac{A^*}{z-\frac{1}{2}j} + \frac{B}{z-j}.$$

The constants A and B are found by

$$A = \lim_{z\to-\frac{1}{2}j}(z+\frac{1}{2}j)\,\frac{Y_1(z)}{z} = \lim_{z\to-\frac{1}{2}j}\frac{z^2-1}{(z-\frac{1}{2}j)(z-j)} = \frac{5}{6},$$

$$B = \lim_{z \to j} (z - j)\ \frac{Y_1(z)}{z} = \lim_{z \to j} \frac{z^2 - 1}{z^2 + \frac{1}{4}} = \frac{8}{3},$$

so that

$$Y_1(z) = \frac{5}{6}\left(\frac{z}{z + \frac{1}{2}j} + \frac{z}{z - \frac{1}{2}j}\right) + \frac{8}{3}\frac{z}{z - j},$$

and we find, by table lookup, that

$$\begin{aligned} y_1(n) &= \frac{5}{6}\left(\left(\frac{1}{2}\right)^n e^{-j\frac{\pi}{2}n} + \left(\frac{1}{2}\right)^n e^{j\frac{\pi}{2}n}\right) + \frac{8}{3}e^{j\frac{\pi}{2}n} \\ &= \underbrace{\frac{5}{3}\left(\frac{1}{2}\right)^n \cos(\frac{\pi}{2}n)}_{y_{1,\mathrm{tr}}(n) \to 0 \text{ for } n \to \infty} + \underbrace{\frac{8}{3}e^{j\frac{\pi}{2}n}}_{y_{1,\mathrm{ss}}(n)}. \end{aligned}$$

Similarly, we find for $y_2(n)$:

$$\begin{aligned} \frac{Y_2(z)}{z} &= \frac{z^2 - 1}{(z + \frac{1}{2}j)(z - \frac{1}{2}j)(z + 1)} = \frac{z - 1}{(z + \frac{1}{2}j)(z - \frac{1}{2}j)} \\ &= \frac{A}{z + \frac{1}{2}j} + \frac{A^*}{z - \frac{1}{2}j}. \end{aligned}$$

The constant A is found by

$$A = \lim_{z \to -\frac{1}{2}j} (z + \frac{1}{2}j)\ \frac{Y_2(z)}{z} = \lim_{z \to -\frac{1}{2}j} \frac{z - 1}{z - \frac{1}{2}j} = \frac{1}{2} - j,$$

so that

$$Y_2(z) = (\frac{1}{2} - j)\frac{z}{z + \frac{1}{2}j} + (\frac{1}{2} + j)\frac{z}{z - \frac{1}{2}j}.$$

The inverse $\mathcal{Z}$-transform is found by table lookup, and we conclude that

$$\begin{aligned} y_2(n) &= (\frac{1}{2} - j)\left(\frac{1}{2}\right)^n e^{-j\frac{\pi}{2}n} + (\frac{1}{2} + j)\left(\frac{1}{2}\right)^n e^{j\frac{\pi}{2}n} \\ &= \underbrace{\left(\frac{1}{2}\right)^n \left(\cos(\frac{\pi}{2}n) + 2\sin(\frac{\pi}{2}n)\right)}_{y_{2,\mathrm{tr}}(n) \to 0 \text{ for } n \to \infty}. \end{aligned}$$

Note that $y_{2,\mathrm{ss}}(n) = 0$, which is consistent with the results obtained in part e.