Solutions examination **Random Signal Processing** (IN4309)

Part I: Digital Signal Processing

November 6, 2009
(14:00 - 17:00)

Assignment 1:

a) Since the system is causal, the region of convergence is the exterior of a circle. Hence, the ROC is given by $|z| > a$.

b)

$$H(z) = c\frac{(z-j)(z+j)}{(z-aj)(z+aj)} = c\frac{z^2+1}{z^2+a^2}$$

c) A causal system is BIBO stable if and only if all poles lie inside the unit circle. Since $|a| < 1$ we conclude that the system is indead BIBO stable.

d) ROC $|z| > a$. Hence, we have a causal solution. In that case we have

$$h(n) = \frac{1}{2\pi j}\oint_C H(z)z^{n-1}dz = \frac{1}{2\pi j}\oint_C c\frac{(z^2+1)z^{n-1}}{z^2+a^2}dz,$$

where C is a counterclockwise closed contour in the region of convergence $|z| > a$. For $n > 0$, we have only two poles inside C, so that

$$\begin{aligned}
h(n) &= \operatorname*{Res}_{z=aj} H(z)z^{n-1} + \operatorname*{Res}_{z=-aj} H(z)z^{n-1} \\
&= \lim_{z\to aj}(z-aj)H(z)z^{n-1} + \lim_{z\to -aj}(z+aj)H(z)z^{n-1} \\
&= \lim_{z\to aj} c\frac{(z^2+1)z^{n-1}}{z+aj} + \lim_{z\to -aj} c\frac{(z^2+1)z^{n-1}}{z-aj} \\
&= c\frac{1-a^2}{2aj}(aj)^{(n-1)} + c\frac{1-a^2}{-2aj}(-aj)^{(n-1)} \\
&= c\frac{a^2-1}{2a^2}(aj)^n + c\frac{a^2-1}{2a^2}(-aj)^n \\
&= c\frac{a^2-1}{2a^2}a^n e^{j\frac{\pi}{2}n} + c\frac{a^2-1}{2a^2}a^n e^{-j\frac{\pi}{2}n} \\
&= c\frac{a^2-1}{a^2}a^n\cos(\frac{\pi}{2}n).
\end{aligned}$$

For $n = 0$, we can apply the initial value theorem for causal systems which states that

$$h(0) = \lim_{z\to\infty} H(z) = c.$$

Alternatively, we can find $h(0)$ by contour integration. For $n = 0$, we have poles at $z = 0, z = aj$ and $z = -aj$ so that

$$\begin{aligned} h(0) &= \operatorname*{Res}_{z=0} H(z)z^{-1} + \operatorname*{Res}_{z=aj} H(z)z^{-1} + \operatorname*{Res}_{z=-aj} H(z)z^{-1} \\ &= \lim_{z\to 0} c\frac{z^2+1}{z^2+a^2} + \lim_{z\to aj} c\frac{z^2+1}{z(z+aj)} + \lim_{z\to -aj} c\frac{z^2+1}{z(z-aj)} \\ &= \frac{c}{a^2} + c\frac{1-a^2}{-2a^2} + c\frac{1-a^2}{-2a^2} = c. \end{aligned}$$

Alternative solution: Partial-fraction expansion

$$\frac{H(z)}{z} = c\frac{z^2+1}{z(z^2+a^2)} = \frac{A}{z-aj} + \frac{A^*}{z+aj} + \frac{B}{z}.$$

The constants A and B are found by

$$A = \lim_{z\to aj}(z-aj)\frac{H(z)}{z} = \lim_{z\to aj} c\frac{z^2+1}{z(z+aj)} = c\frac{1-a^2}{-2a^2} = A^*,$$

$$B = \lim_{z\to 0} z\frac{H(z)}{z} = \lim_{z\to 0} c\frac{z^2+1}{z^2+a^2} = \frac{c}{a^2},$$

and we conclude that

$$H(z) = c\frac{a^2-1}{2a^2}\frac{z}{z-aj} + c\frac{a^2-1}{2a^2}\frac{z}{z+aj} + \frac{c}{a^2}.$$

Since the ROC is $|z| > a$, we find by table lookup (see Table 3.3) that

$$\alpha^n u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha}, \quad |z| > \alpha,$$

and

$$\delta(n) \overset{\mathcal{Z}}{\longleftrightarrow} 1 \text{ for all } z,$$

so that

$$\begin{aligned} h(n) &= c\frac{a^2-1}{2a^2}\left((aj)^n + (-aj)^n\right)u(n) + \frac{c}{a^2}\delta(n) \\ &= c\frac{a^2-1}{2a^2}a^n\left(e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}\right)u(n) + \frac{c}{a^2}\delta(n) \\ &= c\frac{a^2-1}{a^2}a^n\cos(\frac{\pi}{2}n) + \frac{c}{a^2}\delta(n) \end{aligned}$$

e) Since $h(0) = c$ we conclude that $c = 1$.

f) The frequency response of the system is given by

$$H(\omega) = c\frac{1 + e^{-2j\omega}}{1 + a^2 e^{-2j\omega}}.$$

The system is a notch filter. Since the system has zeros at $z = j$ and $z = -j$, the magnitude response will be zero at $\omega = \frac{\pi}{2}$ and $\omega = -\frac{\pi}{2}$, respectively. At frequency $\omega = 0$, we have

$$H(0) = c\frac{2}{1 + a^2}.$$

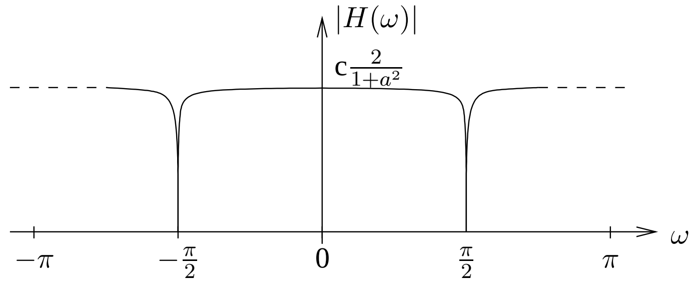

Figure 1: Magnitude response of $H(\omega)$.

Assignment 2:

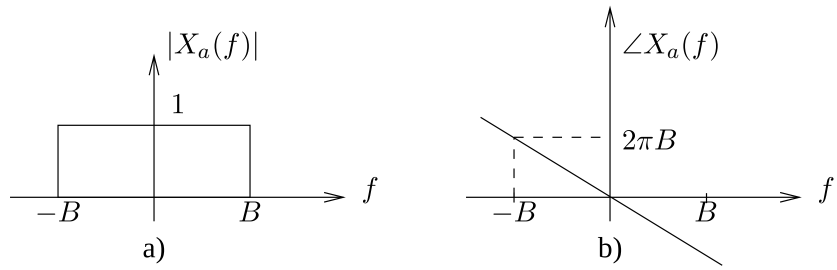

Figure 2: Magnitude a) and phase b) spectrum.

a) The magnitude and phase spectrum are shown in Figure 2a) and b), respectively, so that

$$X_a(f) = \begin{cases} e^{-j2\pi f}, & \text{for } |\omega| \leq B \\ 0, & \text{otherwise} \end{cases} .$$

Therefore,

$$\begin{aligned} x_a(t) &= \int_{-\infty}^{\infty} X_a(f) e^{j2\pi ft} df \\ &= \int_{-B}^{B} e^{j2\pi f(t-1)} df \\ &= \frac{1}{j2\pi(t-1)} \, e^{j2\pi f(t-1)} \Big|_{-B}^{B} \\ &= \frac{1}{j2\pi(t-1)} \left(e^{j2\pi B(t-1)} - e^{-j2\pi f(t-1)} \right) \\ &= \frac{1}{\pi(t-1)} \sin(2\pi B(t-1)). \end{aligned}$$

b) Since $f_{\max} = B$ Hz, we conclude that $f_s > 2B$.

c) The relation between the spectrum of the discrete-time signal x and that of the continuous-time signal x_a is given by

$$X(f) = f_s \sum_{k=-\infty}^{\infty} X_a(f + kf_s).$$

Figure 3 shows the result for $f_s > 2B$.

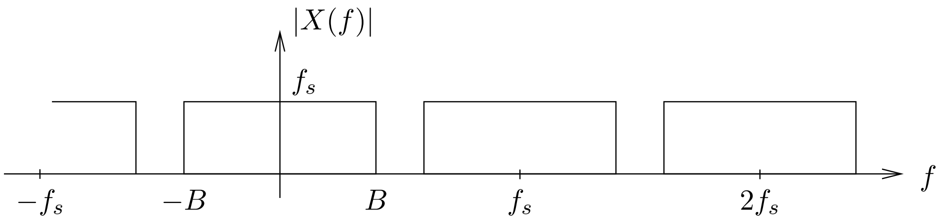

Figure 3: Magnitude spectrum of $X(f)$ ($f_s > 2B$).

d) We have

$$h(n) = h_a(nT_s) = 2\cos(2\pi f_0(nT_s - 1)).$$

Clearly, since $f_s = B < 2B$ we will have aliasing. Ideal interpolation corresponds to ideal low-pass filtering with a cut-off frequency equal to $f_c = f_s/2 = B/2$. In order to find the frequency at which the harmonic will be reconstructed, we have to compute which of the alias frequencies falls within the interval $|f| < B/2$. We have

$$\begin{aligned}\cos(2\pi f_0(nT_s - 1)) &= \cos(-2\pi f_0(nT_s - 1)) \\ &= \cos(2\pi(-f_0 nT_s + f_0)) \\ &= \cos(2\pi((f_s - f_0)nT_s + f_0)).\end{aligned}$$

Since $0 < f_s - f_0 = B - f_0 < B/2$ whenever $B/2 < f_0 < B$, we conclude that

$$\hat{h}_a(t) = 2\cos(2\pi((f_s - f_0)t + f_0)),$$

e) It is not possible to sample the spectrum $X(f)$ and to reconstruct $x(n)$ out of these spectral samples since $x_a(t)$ has an infinite support. The process of sampling $X(f)$ will therefore introduce temporal aliasing.

Assignment 3:

a) Since

$$y(n) = x(n) + x(n-3),$$

we have

$$H(z) = 1 + z^{-3} = \frac{z^3+1}{z^3}.$$

The filter has three (trivial) poles at the origin and three zeros at

$$z_k^3 = -1 = e^{j\pi}e^{j2\pi k} \quad \Rightarrow \quad z_k = e^{j(\frac{\pi}{3}+\frac{2\pi}{3}k)} \;\; k = 0, 1, 2.$$

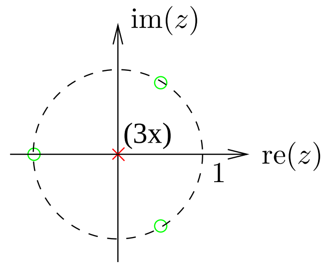

Figure 4: Pole-zero map of $H(z)$.

b) The frequency response is given by

$$H(\omega) = 1 + e^{-j3\omega} = e^{-j\frac{3}{2}\omega}\left(e^{j\frac{3}{2}\omega} + e^{-j\frac{3}{2}\omega}\right) = 2e^{-j\frac{3}{2}\omega}\cos(\frac{3}{2}\omega).$$

Hence the magnitude and phase response are given by

$$|H(\omega)| = 2|\cos(\frac{3}{2}\omega)|,$$

and

$$\angle H(\omega) = -\frac{3}{2}\omega + \underbrace{\angle\cos(\frac{3}{2}\omega)}_{0 \text{ if } \geq 0 \text{ and } \pi \text{ if } < 0}.$$

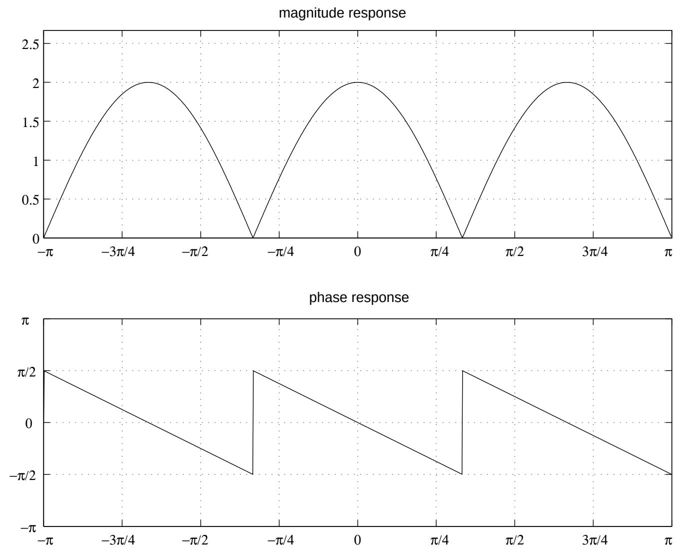

Figure 5: Magnitude (top plot) and phase (bottom plot) response

There are zeros at $\frac{3}{2}\omega = \frac{\pi}{2} \mod \pi$ or $\omega = \frac{\pi}{3} \mod \frac{2\pi}{3}$ (which is consistent with the result obtained in a)).

c) The following input signal

$$x(n) = x_1(n) + x_2(n) + x_3(n) = \delta(n) + e^{j\frac{\pi}{3}n}u(n) + e^{j\frac{2\pi}{3}n}u(n),$$

consists of a unit impulse and two harmonics with frequencies $\omega_2 = \frac{\pi}{3}$ and $\omega_3 = \frac{2\pi}{3}$. The steady-state response of the system is given by

$$y_{\text{ss}}(n) = \lim_{n\to\infty} y(n) = H(\omega_2)x_2(n) + H(\omega_3)x_3(n),$$

since the unit impulse is zero for $n > 0$. Since $H(\omega_2) = H(\frac{\pi}{3}) = 0$ and $H(\omega_3) = H(\frac{2\pi}{3}) = 2$, we conclude that

$$y_{\text{ss}}(n) = 2e^{j\frac{2\pi}{3}n}.$$

d) We have

$$X(z) = X_1(z) + X_2(z) + X_3(z) = 1 + \frac{z}{z - e^{j\frac{\pi}{3}}} + \frac{z}{z - e^{j\frac{2\pi}{3}}},$$

so that

$$Y(z) = H(z)\left(X_1(z) + X_2(z) + X_3(z)\right) = Y_1(z) + Y_2(z) + Y_3(z),$$

where

$$Y_1(z) = H(z)X_1(z) = 1 + z^{-3},$$

$$Y_2(z) = H(z)X_2(z) = \frac{z^3+1}{z^2(z - e^{j\frac{\pi}{3}})},$$

and

$$Y_3(z) = H(z)X_3(z) = \frac{z^3+1}{z^2(z - e^{j\frac{2\pi}{3}})}.$$

Obviously, we have $y_1(n) = \delta(n) + \delta(n-3)$. The inverse $\mathcal{Z}$-transform of the other terms can be computed using, for example, contour integration. For computing $y_2(n)$, we have

$$y_2(n) = \frac{1}{2\pi j}\oint_C \frac{z^3+1}{z - e^{j\frac{\pi}{3}}} z^{n-3} dz,$$

where C is a counterclockwise closed contour in the region of convergence $|z| > 1$. For $n > 2$, we have only one pole inside C, so that

$$\begin{aligned}
y_2(n) &= \operatorname*{Res}_{z=e^{j\frac{\pi}{3}}} Y_2(z)z^{n-1} \\
&= \lim_{z\to e^{j\frac{\pi}{3}}} (z - e^{j\frac{\pi}{3}})Y_2(z)z^{n-1} \\
&= \lim_{z\to e^{j\frac{\pi}{3}}} (z^3+1)z^{n-3} \\
&= (e^{j\pi}+1)e^{j\frac{\pi}{3}(n-3)} = 0.
\end{aligned}$$

For $n = 2$ we have an additional pole at $z = 0$ so that

$$\begin{aligned}
y_2(2) &= \operatorname*{Res}_{z=e^{j\frac{\pi}{3}}} Y_2(z)z + \operatorname*{Res}_{z=0} Y_2(z)z \\
&= \lim_{z\to e^{j\frac{\pi}{3}}} \frac{z^3+1}{z} + \lim_{z\to 0} \frac{z^3+1}{z - e^{j\frac{\pi}{3}}} \\
&= 0 + \frac{1}{-e^{j\frac{\pi}{3}}} = -\frac{1}{2} + j\frac{1}{2}\sqrt{3}.
\end{aligned}$$

For $n = 1$ we have a double pole at the origin so that

$$y_2(1) = \operatorname*{Res}_{z=e^{j\frac{\pi}{3}}} Y_2(z) + \operatorname*{Res}_{z=0} Y_2(z)$$
$$= \lim_{z \to e^{j\frac{\pi}{3}}} \frac{z^3+1}{z^2} + \lim_{z\to 0} \frac{d}{dz}\left(\frac{z^3+1}{z-e^{j\frac{\pi}{3}}}\right)$$
$$= 0 + \lim_{z\to 0} \frac{3z^2(z-e^{j\frac{\pi}{3}}) - (z^3+1)}{(z-e^{j\frac{\pi}{3}})^2}$$
$$= \frac{-1}{e^{j\frac{2\pi}{3}}} = \frac{1}{2} + j\frac{1}{2}\sqrt{3}.$$

The value $y_2(0)$ is easily found by

$$y_2(0) = \lim_{z\to\infty} Y_2(z) = 1.$$

Note that by inspection of the difference equation of the system ($y(n) = x(n) + x(n-3)$), we can conclude that for $n < 3$, the output is simply $x(n)$ ($x(n-3) = 0$ for $n < 3$ since $x(n)$ is causal). Only after $n = 3$, the second term ($x(n-3)$) starts playing a role, thereby creating the zero in the frequency response for the input $x_2(n)$ and a gain of 2 for the input $x_3(n)$. So, we conclude that $y_2(n) = 0$ for $n < 0$, $y_2(n) = x_2(n)$ for $n = 0, 1, 2$ and $y_2(n) = 0$ ($= y_{1,\text{ss}}(n)$) for $n > 2$. Similarly, we find for $y_3(n)$ that $y_3(n) = 0$ for $n < 0$, $y_3(n) = x_3(n)$ for $n = 0, 1, 2$ and $y_3(n) = 2e^{j\frac{2\pi}{3}n}$ ($= y_{2,\text{ss}}(n)$) for $n > 2$.