Solutions Examination **Random Signal Processing** (IN4309)

October 28, 2008
(14:00 - 17:00)

Assignment 1:

a)

$$H(z) = c\frac{(z-a)(z+a)}{(z-aj)(z+aj)} = c\frac{z^2-a^2}{z^2+a^2}, \quad c \in \mathbb{R}.$$

The system is unique up to a constant.

b) Region R_1 specifies the exterior of a circle so that $h(n)$ is causal. Causal LTI systems are BIBO stable if and only if all poles lie *inside* the unit circle. Since $|a| < 1$, we conclude that the system is BIBO stable. When the region of convergence is R_2, the interior of a circle, $h(n)$ is anticausal and we conclude that the system is BIBO stable when all poles lie *outside* the unit circle. Hence, in this case the system is unstable.

c) ROC $|z| > a$. Hence, we have a causal solution. Assume $c = 1$. In that case we have

$$h(n) = \frac{1}{2\pi j}\oint_C H(z)z^{n-1}dz = \frac{1}{2\pi j}\oint_C \frac{(z^2-a^2)z^{n-1}}{z^2+a^2}dz,$$

where C is a counterclockwise closed contour in the region of convergence $|z| > a$. For $n \geq 1$, we have two poles inside C, so that

$$\begin{aligned}
h(n) &= \operatorname*{Res}_{z=aj} H(z)z^{n-1} + \operatorname*{Res}_{z=-aj} H(z)z^{n-1} \\
&= \lim_{z\to aj}(z-aj)H(z)z^{n-1} + \lim_{z\to -aj}(z+aj)H(z)z^{n-1} \\
&= \lim_{z\to aj}\frac{(z^2-a^2)z^{n-1}}{z+aj} + \lim_{z\to -aj}\frac{(z^2-a^2)z^{n-1}}{z-aj} \\
&= (aj)^n + (-aj)^n = a^n e^{j\frac{\pi}{2}n} + a^n e^{-j\frac{\pi}{2}n} = 2a^n\cos(\frac{\pi}{2}n).
\end{aligned}$$

For $n = 0$, we have poles at $z = 0, z = aj$ and $z = -aj$ so that

$$\begin{aligned}
h(0) &= \operatorname*{Res}_{z=0} H(z)z^{-1} + \operatorname*{Res}_{z=aj} H(z)z^{-1} + \operatorname*{Res}_{z=-aj} H(z)z^{-1} \\
&= \lim_{z\to 0}\frac{z^2-a^2}{z^2+a^2} + \lim_{z\to aj}\frac{z^2-a^2}{z(z+aj)} + \lim_{z\to -aj}\frac{z^2-a^2}{z(z-aj)} \\
&= -1 + 1 + 1 = 1.
\end{aligned}$$

Note that the result for $n = 0$ can be found more easily using the initial value theorem for causal sequences, that is,

$$h(0) = \lim_{z \to \infty} H(z) = 1.$$

d) ROC $|z| < a$. Hence, we have an anticausal solution. Assume $c = 1$ and let $p = z^{-1}$. In that case we have

$$h(n) = \frac{1}{2\pi j} \oint_{C'} H(p) p^{-n-1} dp = \frac{-1}{2\pi j} \oint_{C'} \frac{(p^2 - a^{-2}) p^{-n-1}}{p^2 + a^{-2}} dp,$$

where C' is a counterclockwise closed contour in the region of convergence $|p| > a^{-1}$. For $n \leq -1$, we have two poles inside C, so that

$$\begin{aligned}
h(n) &= \operatorname*{Res}_{p=a^{-1}j} H(p)p^{-n-1} + \operatorname*{Res}_{p=-a^{-1}j} H(p)p^{-n-1} \\
&= \lim_{p \to a^{-1}j} (p - a^{-1}j) H(p) p^{-n-1} + \lim_{p \to -a^{-1}j} (p + a^{-1}j) H(p) p^{-n-1} \\
&= \lim_{p \to a^{-1}j} -\frac{(p^2 - a^{-2}) p^{-n-1}}{p + a^{-1}j} - \lim_{p \to -a^{-1}j} \frac{(p^2 - a^{-2}) p^{-n-1}}{p - a^{-1}j} \\
&= -\left(a^{-1}j\right)^{-n} - \left(-a^{-1}j\right)^{-n} \\
&= -a^n e^{-j\frac{\pi}{2}n} - a^n e^{j\frac{\pi}{2}n} = -2a^n \cos(\frac{\pi}{2}n).
\end{aligned}$$

This result could be obtained directly from c) since

$$h(n) = \frac{-1}{2\pi j} \oint_{C'} \frac{(p^2 - a^{-2}) p^{-n-1}}{p^2 + a^{-2}} dp,$$

is exactly the same expression as we found in c) except for a minus sign and we have to replace a by a^{-1} and n by $-n$. Hence we have

$$h(n) = -2(a^{-1})^{(-n)} \cos(\frac{\pi}{2}(-n)) = -2a^n \cos(\frac{\pi}{2}n).$$

Similarly, for $n = 0$ we then have $h(0) = -1$. This result could also be obtained from the initial value theorem for anticausal sequences

$$h(0) = \lim_{z \to 0} H(z) = -1,$$

or via contour integration where for $n = 0$, $H(p)$ has three poles at $p = 0, p = a^{-1}j$ and $p = -a^{-1}j$.

Alternative solution: Partial-fraction expansion

$$\frac{H(z)}{z} = \frac{z^2 - a^2}{z(z^2 + a^2)} = \frac{A}{z - aj} + \frac{A^*}{z + aj} + \frac{B}{z}.$$

The constants A and B are found by

$$A = \lim_{z \to aj} (z - aj) \frac{H(z)}{z} = \lim_{z \to aj} \frac{z^2 - a^2}{z(z + aj)} = 1,$$

$$B = \lim_{z \to 0} z \frac{H(z)}{z} = \lim_{z \to 0} \frac{z^2 - a^2}{z^2 + a^2} = -1,$$

and we conclude that

$$H(z) = \frac{z}{z - aj} + \frac{z}{z + aj} - 1.$$

c) ROC $|z| > a$. By table lookup (see Table 3.3, p. 170) we find that

$$\alpha^n u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - \alpha z^{-1}} = \frac{z}{z - \alpha}, \quad |z| > \alpha,$$

so that

$$\begin{aligned} h(n) &= \left((aj)^n + (-aj)^n\right) u(n) - \delta(n) \\ &= \left(a^n e^{j\frac{\pi}{2}n} + a^n e^{-j\frac{\pi}{2}n}\right) u(n) - \delta(n) \\ &= 2a^n \cos(\frac{\pi}{2}n) u(n) - \delta(n). \end{aligned}$$

d) ROC $|z| < a$. By table lookup we find that

$$-\alpha^n u(-n-1) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - \alpha z^{-1}} = \frac{z}{z - \alpha}, \quad |z| < \alpha,$$

so that

$$\begin{aligned} h(n) &= \left(-(aj)^n - (-aj)^n\right) u(-n-1) - \delta(n) \\ &= -2a^n \cos(\frac{\pi}{2}n) u(-n-1) - \delta(n). \end{aligned}$$

Assignment 2:

a)

$$X_a(f) = \int_0^\infty \alpha^t e^{-j2\pi ft} dt, \quad \alpha < 1.$$

Since $\alpha^t = e^{\ln(\alpha^t)} = e^{t\ln\alpha}$ we have

$$\begin{aligned} X_a(f) &= \int_0^\infty e^{-(j2\pi f - \ln\alpha)t} dt \\ &= \frac{-1}{j2\pi f - \ln\alpha} \; e^{-(j2\pi f - \ln\alpha)t}\Big|_0^\infty \\ &= \frac{1}{j2\pi f - \ln\alpha}. \end{aligned}$$

b)

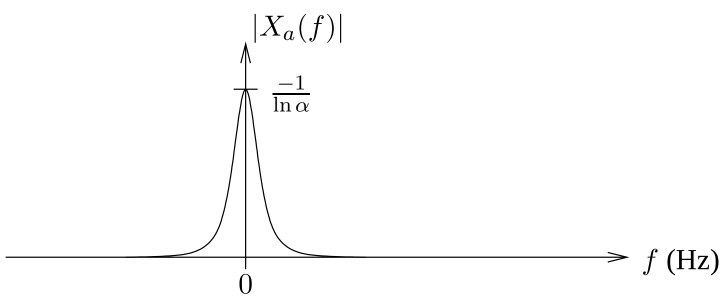

Figure 1: Magnitude spectrum of $X_a(f)$.

c)

$$x(n) = x_a(nT_s) = \alpha^{nT_s} = \left(\alpha^{T_s}\right)^n,$$

so that $\beta = \alpha^{T_s}$.

d)

$$X(f') = \frac{1}{1-\beta e^{-j2\pi f'}} = \frac{1}{1-\alpha^{T_s} e^{-j2\pi f'}}$$
$$= \frac{1}{1-e^{-(j2\pi f' - T_s \ln \alpha)}}.$$

In terms of the frequency $f = f' f_s$, this becomes

$$X(f) = \frac{1}{1-e^{-\frac{j2\pi f - \ln \alpha}{f_s}}}.$$

e)

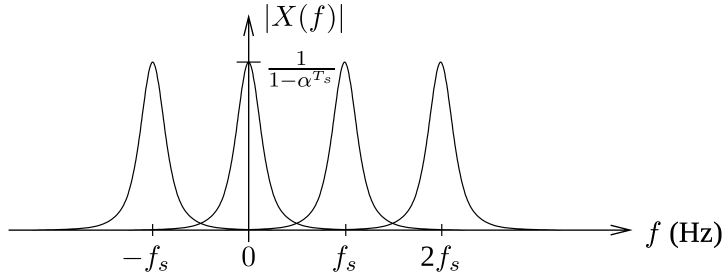

Figure 2: Magnitude spectrum of $X(f)$.

Note that when $f_s \gg 1$, and thus $T_s \ll 1$, we can accurately approximate the term α^{T_s} by its first-order Taylor series expansion given by

$$\alpha^{T_s} \approx 1 + T_s \ln \alpha,$$

so that

$$X(0) = \frac{1}{1-\alpha^{T_s}} \approx \frac{1}{1-(1+T_s \ln \alpha)} = \frac{-f_s}{\ln \alpha} = f_s X_a(0),$$

as expected.

f) Since $X_a(f)$ is *not* bandlimited, we cannot recover $x_a(t)$ out of its samples $x(n)$.

g) Since $x_a(t)$ has infinite support, $x_a(t)$ cannot be recoverd from its periodic extension so that it is *not* possible to recover $X(f)$ out of its samples $X(\frac{2\pi}{N}k)$.

Assignment 3:

a)

$$H(z) = \frac{z-1}{z-a} = \frac{1-z^{-1}}{1-az^{-1}} = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}}.$$

Hence, $b_0 = 1, b_1 = -1$ and $a_1 = -a$.

b) We have a zero at $z = 1$ and a pole at $z = a$.

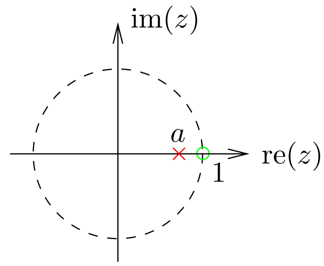

Figure 3: Pole-zero map of $H(z)$.

c) Since the system is causal, the system is BIBO stable if and only if the pole $z = a$ lies inside the unit circle. Hence, $|a| < 1$.

d) Since the system has a zeros at $z = 1$, the magnitude response will be zero at $\omega = 0$. It reaches a maximum at frequency $\omega = \pi$ of

$$\frac{|e^{j\pi} - 1|}{|e^{j\pi} - a|} = \frac{|-1-1|}{|-1-a|} = \frac{2}{1+a}.$$

Note that in case $a \approx 1$, we have

$$|H(\omega)| = \frac{|e^{j\omega} - 1|}{|e^{j\omega} - a|} \approx 1,$$

for ω not too close to zero. The phase response is given by

$$\angle H(\omega) = \angle(e^{j\omega} - 1) - \angle(e^{j\omega} - a).$$

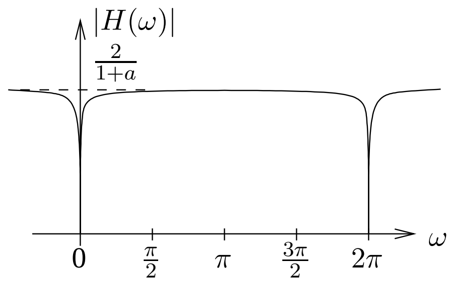

Figure 4: Magnitude response of $H(\omega)$.

We have at $\omega = 0^+$ that

$$\angle H(0^+) = \frac{\pi}{2} - 0 = \frac{\pi}{2},$$

and at $\omega = 0^-$

$$\angle H(0^-) = -\frac{\pi}{2} - 0 = -\frac{\pi}{2}.$$

At $\omega = \pi$ we have

$$\angle H(0) = \pi - \pi = 0.$$

Note that in case $a \approx 1$, we have

$$\angle H(\omega) = \angle(e^{j\omega} - 1) - \angle(e^{j\omega} - a) \approx 0,$$

for ω not too close to zero.

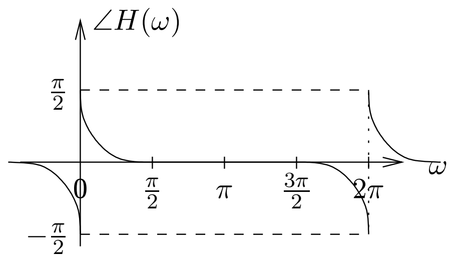

Figure 5: Magnitude response of $H(\omega)$.

e) The impulse impulse is obtained by inverse $\mathcal{Z}$-transformation. This can be done, for example, using contour integration:

$$h(n) = \frac{1}{2\pi j}\oint_C H(z) z^{n-1} dz = \frac{1}{2\pi j}\oint_C \frac{(z-1)z^{n-1}}{z-a} dz,$$

where C is a counterclockwise closed contour in the region of convergence $|z| > a$. For $n \geq 1$ we have one pole at $z = a$ so that

$$h(n) = \operatorname*{Res}_{z=a} H(z) z^{n-1} = \lim_{z\to a} (z-1) z^{n-1} = (a-1)a^{n-1},$$

and $h(0)$ is most easily found by the initial value theorem

$$h(0) = \lim_{z\to\infty} H(z) = 1.$$

Alternatively, we could have solved this problem using partial-fraction expansion. In that case we have

$$\frac{H(z)}{z} = \frac{z-1}{z(z-a)} = \frac{A}{z} + \frac{B}{z-a},$$

where

$$A = \lim_{z\to 0} z\frac{H(z)}{z} = \lim_{z\to 0} \frac{z-1}{z-a} = a^{-1},$$

$$B = \lim_{z\to a} (z-a)\frac{H(z)}{z} = \lim_{z\to a} \frac{z-1}{z} = 1 - a^{-1},$$

so that

$$H(z) = a^{-1} + (1-a^{-1})\frac{z}{z-a}.$$

The inverse transform is then obtained by table lookup (see Table 3.3, p. 170):

$$h(n) = a^{-1}\delta(n) + (1-a^{-1})a^n u(n).$$

f) We have

$$Y(z) = H(z)X(z),$$

with (see Table 3.3, p. 170)

$$X(z) = \frac{z}{z-1}.$$

The inverse transformation can be found by, for example, partial fraction expansion:

$$\frac{Y(z)}{z} = \frac{z-1}{z-a} \cdot \frac{1}{z-1} = \frac{1}{z-a},$$

and we conclude that

$$Y(z) = \frac{z}{z-a}, \quad |z| > a \quad \overset{\mathcal{Z}}{\longleftrightarrow} \quad y(n) = a^n u(n).$$

Since $y(n) = y_{\mathrm{tr}}(n) + y_{\mathrm{ss}}(n)$, with

$$y_{\mathrm{ss}} = \lim_{n \to \infty} y(n) = 0,$$

we conclude that $y_{\mathrm{tr}}(n) = a^n u(n)$. This is to be expected since $x(n) = u(n)$ is a suddenly applied signal with frequency $\omega = 0$ (DC component). Since the system has a zero at $z = 1$, the magnitude response has a zero at $\omega = 0$, and we conlcude that the steady-state response $y_{\mathrm{ss}}(n)$ will be zero.

g) The zero-input response $y_{\mathrm{zi}}(n)$ in this case is zero since the system is initially in rest. Hence $y_{\mathrm{zs}}(n) = y(n) = a^n u(n)$.