TU Delft Mekelweg 4
Faculteit EWI, DIAM 2628 CD Delft

Exam Measure and Integration Theory (wi4320)
April 11, 2013, 14.00-17.00

Grading: $\frac{1}{10}((20) + (10 + 10) + (10 + 10 + 5) + (5 + 5 + 5 + 5 + 5) + (10 \text{ free}))$

Motivate your solutions!

1. Let $(X, \mathscr{M}, \mu)$ be a measure space and let $1 \leqslant p_0 \leqslant p_1 \leqslant \infty$. For $0 < \theta < 1$ define the number $1 \leqslant p \leqslant \infty$ by the relation
$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$
Prove that if $f \in L^{p_0}(X, \mu) \cap L^{p_1}(X, \mu)$, then $f \in L^{p_\theta}(X, \mu)$ and
$$\|f\|_{p_\theta} \leqslant \|f\|_{p_0}^{1-\theta} \|f\|_{p_1}^{\theta}.$$
Hint: Apply Hölder's inequality.

2. The Dirac measure δ_0 on $\mathbb{R}$ is the Borel measure defined by $\delta(B) = 1$ if $0 \in B$ and $\delta(B) = 0$ if $0 \notin B$.

   (a) Show that every Borel function $f : \mathbb{R} \to \mathbb{R}$ is integrable with respect to δ_0 and
   $$\int_{\mathbb{R}} f \, d\delta_0 = f(0).$$
   Hint: Show first that $f = \mathbf{1}_{\{0\}}$ δ_0-almost everywhere.

   (b) Show that δ_0 is singular with respect to the Lebesgue measure of $\mathbb{R}$.

3. A Borel probability measure μ on $\mathbb{R}$ is called *regular* if for all Borel subsets B of $\mathbb{R}$ and all $\varepsilon > 0$ there is a closed set F in $\mathbb{R}$ and an open set G in $\mathbb{R}$ such that $F \subseteq B \subseteq G$ and $\mu(G \setminus F) < \varepsilon$.

   Show that every Borel probability measure μ on $\mathbb{R}$ is regular by completing the following steps:

   (a) Show that the collection of all Borel sets B in $\mathbb{R}$ which have the property that for all $\varepsilon > 0$ there exist a closed set F in $\mathbb{R}$ and an open set G in $\mathbb{R}$ such that $F \subseteq B \subseteq G$ and $\mu(G \setminus F) < \varepsilon$ is a σ-algebra.

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(b) Show that this collection contains all closed sets F in $\mathbb{R}$.
Hint: If F is a closed set, consider the open sets

$$G_n = \{x \in \mathbb{R} : \inf\{|x - y| : y \in F\} < 1/n\}$$

and show that $\mu(G_n \setminus F) \to 0$ as $n \to \infty$.

(c) Derive the asserted result from (a) and (b).

4. In this exercise we work over the real scalar field; all functions are assumed to be real-valued. Let $(X, \mathscr{M}, \mu)$ be a finite measure space. We are going to prove that the dual of $L^1(X, \mu)$ is $L^\infty(X, \mu)$.

(a) Show that for every $f \in L^\infty(X, \mu)$ the mapping $\phi_f : L^1(X, \mu) \to \mathbb{R}$, defined by

$$\phi_f(g) := \int_X fg \, d\mu, \qquad g \in L^1(X, \mu),$$

is bounded.

Suppose, conversely, that $\phi : L^1(X, \mu) \to \mathbb{R}$ is a bounded linear functional.

(b) Show that $\nu(B) := \phi(\mathbf{1}_B)$, $B \in \mathscr{M}$, defines a real-valued measure ν on $(X, \mathscr{M})$.

(c) Show that ν is absolutely continuous with respect to μ.

Let f be the Radon-Nikodým derivative of ν with respect to μ.

(d) Show that $f \in L^\infty(X, \mu)$.

Hint: Show that if $f \notin L^\infty(X, \mu)$, then for each $n \geqslant 1$ there exists a set $M \in \mathscr{M}$ such that $|\phi(\mathbf{1}_M)| \geqslant n\|\mathbf{1}_M\|_1$.

(e) Show that $\phi = \phi_f$, where ϕ_f is the bounded linear functional associated with f defined in part (a).

THE END