TU Delft — Mekelweg 4
Faculteit EWI, DIAM — 2628 CD Delft

Exam Measure and Integration theory
January 22, 2013; 14.00 - 17.00

All solutions should be carefully motivated.
Grading: $\frac{1}{10} \times [(5+10+10+5+5)+(5+15)+(10+10+15)+(10 \text{ free})]$

In Problems 1 and 2, $(X, \mathscr{M}, \mu)$ is an arbitrary measure space.

1. Suppose a nonnegative function $f \in L^1(\mu)$ is given.

   (a) Show that $\nu(A) := \int_A f \, d\mu$ defines a measure on $(X, \mathscr{M})$.

   (b) Show that a measurable function g belongs to $L^1(\nu)$ if and only if $fg \in L^1(\mu)$, and that in that case the following identity holds:

   $$\int_X g \, d\nu = \int_X fg \, d\mu.$$

   (c) Show that ν is absolutely continuous with respect to μ. Does ν have a density (Radon-Nikodym derivative) with respect to μ? If yes, determine it; if no, prove this.

   Suppose now that $f \in L^1(\mu)$ is an arbitrary function.

   (d) Show that $\nu(A) := \int_A f \, d\mu$ defines a real-valued measure on $(X, \mathscr{M})$.

   (e) Determine its Hahn decomposition (its decomposition into positive and negative parts) and prove that your answer is indeed correct.

2. Let $1 \leqslant p < \infty$ and suppose a function $f \in L^p(\mu)$ is given.

   (a) Show that the function $r \mapsto \mu(\{|f| > r\})$ is measurable.

   (b) Show that

   $$\int_X |f|^p \, d\mu = p \int_0^\infty r^{p-1} \mu(\{|f| > r\}) \, dr.$$

   Hint: Write $\mu(\{|f| > r\})$ as an integral.
   Do you see a way to generalise this identity?

-- please turn the page --

3. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ be given, and let

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| \, dy, \quad x \in \mathbb{R}^d,$$

be the associated Hardy-Littlewood maximal function. Here, $B_r(x) = \{y \in \mathbb{R}^d : |x-y| < r\}$ is the open ball of radius r centred at x, and $|B_r(x)|$ is its Lebesgue measure.

(a) Show that Mf, as an extended real-valued function, is measurable.

(b) Show that, for any function $f \in L^1(\mathbb{R}^d)$, we have $Mf \in L^1(\mathbb{R}^d)$ if and only if $f = 0$ in $L^1(\mathbb{R}^d)$.

Hint: By considering the balls $B_{|2x|}(x)$ with $|x|$ sufficiently large, first establish that if $f \neq 0$, then there exists a constant $C > 0$ and an $r > 0$ such that $Mf(x) \geqslant C|x|^{-d}$ for all $|x| \geqslant r$.

(c) What can be said about Mf if:

(i) $f \in L^1(\mathbb{R}^d)$;

(ii) $f \in L^p(\mathbb{R}^d)$ with $1 < p < \infty$;

(iii) $f \in L^\infty(\mathbb{R}^d)$.

State the relevant theorems and briefly indicate the main ingredients of the proofs; *no proofs are required.*

`-- end of the exam --`