TU Delft Mekelweg 4
Faculteit EWI, DIAM 2628 CD Delft

Exam Measure and Integration theory
January 27, 2012; 14.00 - 17.00

All solutions should be carefully motivated.
Grading: $(\frac{1}{2}+1+\frac{1}{2}+1)+2+2+2+(1\ \texttt{free})$

1. Let X be a non-empty set and let $\mathscr{A}$ be the collection of all subsets of X that are countable or have countable complement.

   (a) Prove that $\mathscr{A}$ is a σ-algebra.

   (b) Prove that $\mathscr{A}$ is generated by the collection of all singletons $\{x\}$, $x \in X$.

   Define $\tau : \mathscr{A} \to [0,\infty]$ by

   $$\tau(A) = \begin{cases} n, & \text{if } A \text{ contains } n \text{ elements;} \\ \infty & \text{if } A \text{ contains infinitely many elements.} \end{cases}$$

   (c) Prove that τ is a measure on $(X,\mathscr{A})$. When is this measure σ-finite?

   Let $f : X \to \mathbb{R}$ be a function and let $A \subseteq X$ be a countable subset. Suppose that $f \equiv 0$ on the complement of A and that $\sum_{a\in A} |f(a)| < \infty$.

   (d) Prove that f is integrable with respect to τ and

   $$\int_X f\,d\tau = \sum_{a\in A} f(a).$$

2. Prove that a function $f : [0,1] \to [0,1]\times[0,1]$ is Borel measurable if and only if the coordinate functions $\pi_1 \circ f : [0,1] \to [0,1]$ and $\pi_2 \circ f : [0,1] \to [0,1]$ are Borel measurable. Here, $\pi_1(x_1,x_2) = x_1$ and $\pi_2(x_1,x_2) = x_2$ are the coordinate projections.

3. Let $1 \leqslant p < q \leqslant \infty$. Show that if $f \in L^p(\mathbb{R}) \cap L^q(\mathbb{R})$, then for all $p < r < q$ we have $f \in L^r(\mathbb{R})$.

   Hint: Write $f = f1_{\{|f|\leqslant 1\}} + f1_{\{|f|>1\}}$.

4. Let $(X,\mathscr{A},\mu)$ be a measure space and let $f : X \to \mathbb{R}$ be integrable. Show that for all $\varepsilon > 0$ there is a $\delta > 0$ such that $\int_A f\,d\mu < \varepsilon$ for all $A \in \mathscr{A}$ satisfying $\mu(A) < \delta$.

-- please turn the page --

5. (Bonus problem for 1 extra point). On $[0,1] \times [0,1]$ we consider the product σ-algebra $\mathscr{A} \otimes \mathscr{B}$, where $\mathscr{A}$ is the countable/co-countable σ-algebra of Problem 1 and $\mathscr{B}$ the Borel σ-algebra. Is the diagonal

$$\Delta = \{(x,x) : \ x \in [0,1]\}$$

measurable in this σ-algebra?

Hint: Intersect with $[0,1] \times [0, \frac{1}{2}]$ an consider the mapping $x \mapsto (x,x)$. Figure out yourself how to use this hint!

`-- end of the exam --`