Exam Measure and Integration Theory
January 18, 2010

Please motivate your answers!

1. a) Prove that every non-decreasing function $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable.
   b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ have the properties
      i) for all $x \in \mathbb{R}$ the function $y \mapsto f(x, y)$ is non-decreasing;
      ii) for all $y \in \mathbb{R}$ the function $x \mapsto f(x, y)$ is non-decreasing.

   Prove or disprove: f is Borel measurable.

2. For a finite measure λ on a measurable space $(X, \mathscr{A})$ we define

$$\|\lambda\| := \sup_{f \in B(X),\ \|f\|_\infty \le 1} \Big| \int_X f \, d\lambda \Big|,$$

   where $B(X)$ is the Banach space of all bounded measurable functions $f : X \to \mathbb{R}$ endowed with the norm $\|f\|_\infty := \sup_{x \in \mathbb{R}} |f(x)|$.

   a) Compute $\|\delta_x\|$, where δ_x is the Dirac measure concentrated at $x \in X$.
   b) Show that if μ is a measure on $(X, \mathscr{A})$ and ν is the finite measure on $(X, \mathscr{A})$ defined by

$$\nu(A) := \int_A f \, d\mu \qquad (A \in \mathscr{A}),$$

   where $f \in L^1(\mu)$ is non-negative, then $\|\nu\| = \|f\|_{L^1(\mu)}$.
   c) Show that for all finite measures λ_1 and λ_2 on $(X, \mathscr{A})$ we have

$$\|\lambda_1 + \lambda_2\| = \|\lambda_1\| + \|\lambda_2\|,$$

   where $\lambda_1 + \lambda_2$ is the finite measure defined by $(\lambda_1 + \lambda_2)(A) := \lambda_1(A) + \lambda_2(A)$ for $A \in \mathscr{A}$.

3. Let $k : [0, 1] \times [0, 1] \to \mathbb{R}$ be a bounded measurable function with the property that for almost all $t \in [0, 1]$ the function $s \mapsto k(s, t)$ is continuous. Let $f \in L^1(0, 1)$ be given.

   (a) Show that for all $s \in [0, 1]$ the function $t \mapsto k(s, t)f(t)$ is integrable.
   (b) Prove, using the dominated convergence theorem, that the function $g : [0, 1] \to \mathbb{R}$ defined by

$$g(s) := \int_0^1 k(s, t) f(t) \, dt$$

   is continuous. *Hint:* Argue via sequential continuity.

--- Please turn the page ---

4. Prove, using Fubini's theorem, that if $(X, \mathscr{A}, \mu)$ is a probability space, then for all $1 \leq p < \infty$ and $f \in L^p(\mu)$ we have

$$\int_X |f|^p \, d\mu = p \int_0^\infty t^{p-1} \mu(\{|f| > t\}) \, dt.$$

5. Consider a probability space $(\Omega, \mathscr{A}, \mathbb{P})$ and let $f \in L^1(\mathbb{P})$ be given. Let $A \in \mathscr{A}$ be a given set.

   a) Give an explicit description of the σ-algebra $\sigma(A)$ generated by A.

   b) Compute the conditional expectation $\mathbb{E}(f|\sigma(A))$.

   Let $\mathscr{B}$ and $\mathscr{C}$ be two sub$-\sigma$-algebras of $\mathscr{A}$.

   c) Prove that if $\mathscr{B} \subseteq \mathscr{C}$, then

$$\mathbb{E}(\mathbb{E}(f|\mathscr{B})|\mathscr{C}) = \mathbb{E}(\mathbb{E}(f|\mathscr{C})|\mathscr{B}) = \mathbb{E}(f|\mathscr{B}).$$

--- The end ---

Grading: $[(2+4) \ + \ (2+3+4) + (2+4) + (5) + (2+4+4+) + 4 \text{ free})]/4$