Delft University of Technology — Mekelweg 4
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Financial Mathematics WI4079
First Partial Exam
October 31st 2013, 9.30-12.30

Please provide your student card on the table, ready for inspection.
Mobiles, tablets and similar objects must be switched off.
During the first hour you cannot leave the room, even if you decide not to hand your exam in.
After the first hour, if you need to go to the toilet (max one person at a time) you have to temporarily hand your exam and mobile in.
The exam is invalidated if you cheat, use your mobile, etc.
Please write with a pen. Pencils are not accepted.
Please write your name, surname and student number. Do the same for any additional sheet of paper.

Complete solutions will be on Blackboard in the next days.

Good Luck!

Name: .... **Surname:** .... **Student Number:** ....

Questions

Please: Write clearly and try to be as complete as possible.
The different questions give different points. You can find the actual values in the brackets.
The total amount of points is 20.

1. Let $\mathcal{A}$ be the class of all the subsets of $\mathbb{R}$, which are finite unions of disjoint semi-open intervals like $(a, b]$ and $(c, +\infty)$. Enrich $\mathcal{A}$ by including the set $\emptyset$. Show that $\mathcal{A}$ is closed under finite intersections. (2 Points)

2. Let $\mathcal{A}$ be defined as in the text of the previous point. Consider all the intervals of type $(a, b]$ and $(c, +\infty)$ in $\mathbb{R}$. Let $F : \mathbb{R} \to [0, 1]$ be a proper repartition function. Then define the following probabilities:
$$P_0((a, b]) = F(b) - F(a) \qquad \text{and} \qquad P_0((c, +\infty)) = 1 - F(c),$$
and
$$P_1(A) := \sum_{i=1}^{n} P_0((a_i, b_i]),$$
for $A \in \mathcal{A}$.
Show that P_1 is a probability on $\mathcal{A}$. (3 Points)

3. Let us assume that, for every $t_1, ..., t_n \in T$, $n \geq 1$, we have defined $P^{t_1,\dots,t_n}$ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Let also assume that the following compatibility conditions hold:
   a. $P^{t_1,\dots,t_n}(A_1 \times ... \times A_n) = P^{t_{s_1},\dots,t_{s_n}}(A_{s_1} \times ... \times A_{s_n})$, where $(s_1, ..., s_n)$ is any permutation of $(1, ..., n)$, $n \geq 1$.
   b. $P^{t_1,\dots,t_n}(A \times \mathbb{R}) = P^{t_1,\dots,t_{n-1}}(A)$, $\forall A \in \mathcal{B}(\mathbb{R}^{n-1})$.

   Now let us define P on $(\mathbb{R}^T, \mathcal{B}(\mathbb{R}^T))$ such that
$$P(I_{t_1,\dots,t_n}(A)) = P\left((x_t)_{t\in T} \in \mathbb{R}^T : (x_{t_1}, ..., x_{t_n}) \in A\right) := P^{t_1,\dots,t_n}(A), \qquad A \in \mathcal{B}(\mathbb{R}^n),$$
where $I_{t_1,\dots,t_n}(A)$ is a cylinder of basis A.
Show that P is well-defined. (2 Points)
(Hint: Assume that $I_{t_{i_1},\dots,t_{i_n}}(A) = I_{t_{j_1},\dots,t_{j_m}}(B)$, with $A \in \mathcal{B}(\mathbb{R}^n)$ and $B \in \mathcal{B}(\mathbb{R}^m)$.)

4. Let $\{B(t, \omega)\}_{t\geq 0}$ be a family of random variables, which are defined on a probability space $(\Omega, \mathcal{F}, P)$, with values in $\mathbb{R}$. When is the process $B(t, \omega)$ a Brownian motion? (2 Points)

5. Let $B(t,\omega)$ be a Brownian motion:

- Show that $-B(t,\omega)$ is a Brownian motion as well. (1 Point)
- Show that also $tB(1/t,\omega)$, for $t > 0$, is a Brownian motion. (1 Point)

In both cases you can skip the part of the proof related to the almost surely continuous trajectories.

6. Show that a Brownian motion is a martingale with respect to its natural filtration and the Wiener measure P. (2 Points)

7. Let $B(t,\omega)$ be a Brownian motion; show that the process $\{B^2(t,\omega) - t\}_{t\geq 0}$ is a martingale with respect to the natural filtration of $B(t,\omega)$ and the Wiener measure P. You can assume this new process to be integrable and adapted. (1 Point)
(Hint: start by using the same trick we have used in class to prove the martingality of $B(t,\omega)$. In other terms, add and subtract something...)

8. Consider the simple process $\phi \in M_2[0,T]$. Show that $\int_0^T c\phi(t,\omega)\mathrm{d}B(t,\omega) = c\int_0^T \phi(t,\omega)\mathrm{d}B(t,\omega)$, where c is a constant and $B(t,\omega)$ is a Brownian motion. (1 Point)

9. Describe the procedure we can use to define the Itō's integral of a general function $f \in M_2[0,T]$. (3 Points)

10. Let $a(t)$ and $b(t)$ be two deterministic (i.e. non-random) functions. Now assume that the returns of a given financial product are well approximated by the following stochastic differential

$$\mathrm{d}X(t,\omega) = a(t)\mathrm{d}t + b(t)\mathrm{d}B(t,\omega).$$

Compute $E[X(t,\omega)]$ and $Var[X(t,\omega)]$. (2 Points)
(Note: There are at least 2 ways of solving this exercise. Both are correct.)

Answers

1. From what we have seen in class, we know that $\mathcal{A}$ is an algebra, hence it must be closed under finite intersections. However, to prove this it is sufficient to consider two sets in $\mathcal{A}$, namely $A = \bigcup_{i=1}^n (a_i, b_i]$ and $B = \bigcup_{j=1}^m (a_j^*, b_j^*]$. Then

$$A \cap B = \left(\bigcup_{i=1}^n (a_i, b_i]\right) \cap \left(\bigcup_{j=1}^m (a_j^*, b_j^*]\right) = \bigcup_{i=1}^n \bigcup_{j=1}^m (a_i, b_i] \cap (a_j^*, b_j^*].$$

Since all the intervals $(a_i, b_i]$ and $(a_j^*, b_j^*]$ are disjoint, we have that the intersection $(a_i, b_i] \cap (a_j^*, b_j^*]$ is either empty or an interval of type $(a, b]$, hence we easily derive $A \cap B \in \mathcal{A}$. This simple reasoning can be extended to any finite intersection of elements of $\mathcal{A}$.

2. First notice that this question does not ask you to prove that P_1 is well-defined. Hence we can skip that part.
To prove that P_1 is a probability on $\mathcal{A}$, we have to show the following points

a. $P_1(\mathbb{R}) = 1$;

b. $\forall A \in \mathcal{A}, P_1(A) \geq 0$;

c. For $A_1, A_2, ...A_n \in \mathcal{A}$ pairwise-disjoint, the probability P_1 is (finitely) additive, i.e.

$$P_1\left(\cup_{i=1}^n A_i\right) = \sum_{i=1}^n P_1(A_i).$$

We know that F is a repartition function. In the lecture notes you can find all the properties on page 4. Therefore, for point 1, just set $\mathbb{R} = (-\infty, +\infty)$. Then

$$P_1((-\infty, +\infty)) = \lim_{c\to-\infty} P_1((c, +\infty)) = \lim_{c\to-\infty}(1 - F(c)) = 1.$$

Point 2 is given by the fact that F is non-decreasing, and $P_1((a_i, b_i]) = F(b_i) - F(a_i) \geq 0$.
Finally, for point 3, we know that, given two disjoint sets $A = \bigcup_{i=1}^{n}(a_i, b_i]$ and $B = \bigcup_{j=1}^{m}(a_j^*, b_j^*]$ in $\mathcal{A}$, $A \cup B \in \mathcal{A}$. In particular we can write

$$A \cup B = \left(\bigcup_{i=1}^{n}(a_i, b_i]\right) \cup \left(\bigcup_{j=1}^{m}(a_j^*, b_j^*]\right) = \bigcup_{k=1}^{n+m}(c_k, d_k].$$

Hence

$$P_1(A \cup B) = \sum_{i=1}^{n+m} P_0((c_k, d_k]) = \sum_{i=1}^{n} P_0((a_i, b_i]) + \sum_{j=1}^{m} P_0((a_j^*, b_j^*]) = P_1(A) + P_1(B).$$

3. Assume that $I_{t_{i_1},...,t_{i_n}}(A) = I_{t_{j_1},...,t_{j_m}}(B)$, with $A \in \mathcal{B}(\mathbb{R}^n)$ and $B \in \mathcal{B}(\mathbb{R}^m)$. W.l.o.g. we can assume that both $t_{i_1}, ..., t_{i_n}$ and $t_{j_1}, ..., t_{j_m}$ are subsets of $t_1, ..., t_N$. Hence, using the compatibility conditions:

$$\begin{aligned} P^{t_{i_1},...,t_{i_n}}(A) &= P^{t_1,...,t_N}\left((x_{t_1}, ..., x_{t_N}) : (x_{t_{i_1}}, ..., x_{t_{i_n}}) \in A\right) \\ &= P^{t_1,...,t_N}\left((x_{t_1}, ..., x_{t_N}) : (x_{t_{j_1}}, ..., x_{t_{j_m}}) \in B\right) = P^{t_{j_1},...,t_{j_m}}(B) \end{aligned}$$

4. Here we simply have to give the definition of Brownian motion. In other words, $B(t)$ is a Brownian motion if

- For $0 = t_0 < t_1 < ... < t_n < +\infty$, the increments $B(t_1), B(t_2) - B(t_1), ..., B(t_n) - B(t_{n-1})$ are independent random variables;
- Each increment $B(t) - B(s)$, $s < t$, follows a normal distribution with mean 0 and variance $t - s$, i.e. $E[B(t) - B(s)] = 0$ and $Var[B(t) - B(s)] = t - s$;
- The trajectories $t \to B(t)$ are continuous almost surely, that is to say with unitary probability.

5. There are different possibilities to prove these two points. For example we can use the fact that the increments are normally distributed and play with that. We can simply notice that $E[B(t)] = 0$ for $t \geq 0$, and $Cov[B(t), B(s)] = Cov[B(s) + (B(t) - B(s)), B(s)] = Cov[B(s), B(s)] + Cov[B(t) - B(s), B(s)] = Var(B(s)) = s$, for $0 \leq s < t$. Hence

- For $t \geq 0$, $E[-B(t)] = -E[B(t)] = 0$, and for $0 \leq s < t$, $Cov[-B(t), -B(s)] = Cov[B(t), B(s)] = s$;
- First notice that $1/t$ is nothing more than a time rescaling.
  For $t \geq 0$, $E[tB(1/t)] = tE[B(1/t)] = 0$, and for $0 \leq s < t$, $Cov[tB(1/t), sB(1/s)] = stCov[B(1/t), B(1/s)] = st\frac{1}{t} = s$;

The fact that $B(t)$ is a Gaussian process does the rest.

6. The answer to this point is Proposition 7 on page 25 in the lecture notes. In particular it is nice to remember the trick of writing $B(t) = B(t) + B(s) - B(s)$ with $s \leq t$, since we can use it in the next point.

7. Since we do not have to prove adaptivity and integrability, we can simply prove the actual martingale part, i.e.

$$E[B^2(t) - t|\mathcal{F}_s] = B^2(s) - s, \qquad 0 \leq s \leq t.$$

The starting point is always the same trick of adding and subtracting $B(s)$, that is

$$B^2(t) - t = (B(s) + (B(t) - B(s)))^2 - t = B^2(s) + (B(t) - B(s))^2 + 2B(s)(B(t) - B(s)) - t.$$

At this point, let us notice that

$$\begin{aligned} &E[B^2(s)|\mathcal{F}_s] = B^2(s), \\ &E[2B(s)(B(t) - B(s))|\mathcal{F}_s] = 2B(s)E[(B(t) - B(s))|\mathcal{F}_s] = 2B(s)E[(B(t) - B(s))] = 0, \\ &E[(B(t) - B(s))^2|\mathcal{F}_s] = E[(B(t) - B(s))^2] = t - s. \end{aligned}$$

Hence

$$E[B^2(t) - t|\mathcal{F}_s] = B^2(s) + 0 + t - s - t = B^2(s) - s.$$

8. We simply use the definition of Itō's integral for a simple process, i.e.

$$\int_0^T \phi(t,\omega)\mathrm{d}B(t,\omega) := \sum_{j=0}^{n-1} a_j(\omega)\left[B(t_{j+1},\omega) - B(t_j,\omega)\right].$$

Hence

$$\begin{aligned}\int_0^T c\phi(t,\omega)\mathrm{d}B(t,\omega) &= \sum_{j=0}^{n-1} ca_j(\omega)\left[B(t_{j+1},\omega) - B(t_j,\omega)\right] \\ &= c\sum_{j=0}^{n-1} a_j(\omega)\left[B(t_{j+1},\omega) - B(t_j,\omega)\right] = c\int_0^T \phi(t,\omega)\mathrm{d}B(t,\omega).\end{aligned}$$

9. In order to define the Itō's integral for $f \in M_2[0,T]$ we schematically follow the steps below:

- We introduce the concept of simple process ϕ and we define the corresponding Itō's integral, in a way which recalls standard calculus (here we have random rectangles, but the idea is more or less the same);
- We introduce the concept of approximating sequence for a function $f \in M_2$;
- We show that every continuous and bounded function $g \in M_2$ can be approximated by a proper sequence of simple processes $\{\phi_n\}$;
- We then show that every bounded function $h \in M_2$ can be approximated by a proper sequence of continuous and bounded functions $\{g_n\}$;
- Hence we show that every function $f \in M_2$ can be approximated by a proper sequence of bounded functions $\{h_n\}$, that is to say by a proper sequence of simple processes $\{\phi_n\}$, which we can find both in g and h;
- Finally we define $\int_0^T f(t,\omega)\mathrm{d}B(t,\omega)$ as the limit of $\int_0^T \phi_n(t,\omega)\mathrm{d}B(t,\omega)$, for $\phi_n \to f$ as $n \to \infty$.

Naturally, we could improve this short answer by adding details, proofs, etc. Here we just give the basic ingredients of a correct answer, meaning that all these elements must be cited.

10. We know that

$$\mathrm{d}X(t,\omega) = a(t)\mathrm{d}t + b(t)\mathrm{d}B(t,\omega).$$

This means that

$$X(t,\omega) = X(0) + \int_0^t a(s)\mathrm{d}s + \int_0^t b(s)\mathrm{d}B(s,\omega).$$

Clearly

$$E[X(t,\omega)] = X(0) + \int_0^t a(s)\mathrm{d}s,$$

since we know that $E[\int_0^t b(s)\mathrm{d}B(s,\omega)] = 0$.
For what concerns the variance, on the contrary,

$$Var[X(t,\omega)] = E\left[\left(\int_0^t b(s)\mathrm{d}B(s,\omega)\right)^2\right] = E\left[\int_0^t b^2(s)\mathrm{d}s\right] = \int_0^t b^2(s)\mathrm{d}s.$$

The result about the variance can also be obtained using Itō's formula, by setting $g(t,x) = x^2$, $g'_t = 0$, $g'_x = 2x$ and $g''_{xx} = 2$. The formula allows us to compute $X^2(t,\omega)$, hence $E[X^2(t,\omega)]$. This last quantity can be used in the variance decomposition to compute the variance, once we know the expected value $E[X(t,\omega)]$. In fact we know that $Var[X] = E[X^2] - (E[X])^2$.