Exam 2009-2010

1. Consider the Vasicek interest rate model

$$\mathrm{d}R(t) = (\alpha - \beta R(t))\,\mathrm{d}t + \sigma\,\mathrm{d}W(t),$$

where $\{W(t); t \geq 0\}$ is Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and α, β and σ are positive constants. Find a probability measure $\mathbb{Q}$ such that $e^{\beta t}R(t)$ is a $\mathbb{Q}$-martingale.

Solution:

$$\begin{aligned} \mathrm{d}\, e^{\beta t}R(t) &= \beta e^{\beta t}R(t)\mathrm{d}\,t + e^{\beta t}\mathrm{d}\,R(t) \\ &= \sigma e^{\beta t}\left[\frac{\alpha}{\sigma}\,\mathrm{d}t + \mathrm{d}\,W(t)\right] \\ &= \sigma e^{\beta t}\mathrm{d}\,\widetilde{W}(t) \end{aligned}$$

where

$$\widetilde{W}(t) = W(t) + \frac{\alpha}{\sigma}t.$$

Define

$$\mathbb{Q}(A) = \int_A \exp\left[-\frac{\alpha}{\sigma}W(T) - \frac{1}{2}\left(\frac{\alpha}{\sigma}\right)^2 T\right]\mathrm{d}\mathbb{P}, \quad A \in \mathcal{F}_T.$$

By Girsanov's Theorem, $\{\widetilde{W}(t), t \in [0,T]\}$ is a Brownian motion with respect to $\mathbb{Q}$. Since

$$\mathbb{E}_{\mathbb{Q}}\int_0^T \left(\sigma e^{\beta t}\right)^2\,\mathrm{d}t = \frac{\sigma^2}{2\beta}\left(e^{2\beta T} - 1\right) < \infty,$$

it follows from the properties of the stochastic integral that $\{e^{\beta t}R(t); t \in [0,T]\}$ is a $\mathbb{Q}$-martingale.

2. Consider the Black-Scholes model for the asset price

$$\mathrm{d}S(t) = \alpha S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}W(t)$$

where $\{W(t); t \geq 0\}$ is Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and α and σ are positive constants. The interest rate of the bank account is a constant $r > 0$.

   (a) Give the definition of the risk-neutral probability $\widetilde{\mathbb{P}}$ and its Radon-Nikodým derivative with respect to $\mathbb{P}$.

   Solution: A probability measure $\widetilde{\mathbb{P}}$ is said to be risk-neutral if

   i. $\widetilde{\mathbb{P}} \equiv \mathbb{P}$,

   ii. discounted stock price $e^{-rt}S(t)$ is a martingale.

   A random variable Z is the Radon-Nikodým derivative of $\widetilde{\mathbb{P}}$ with respect to $\mathbb{P}$ if for all $A \in \mathcal{F}$

$$\widetilde{\mathbb{P}}(A) = \int_A Z\,\mathrm{d}\mathbb{P}.$$

(b) Derive the risk-neutral pricing formula.

Solution: The differential of the discounted stock price is given by

$$\begin{aligned} \mathrm{d}e^{-rt}S(t) &= -re^{-rt}S(t)\,\mathrm{d}t + e^{-rt}\,\mathrm{d}S(t) \\ &= \sigma e^{-rt}S(t)\left[\frac{\alpha - r}{\sigma}\,\mathrm{d}t + \mathrm{d}W(t)\right]. \end{aligned}$$

Define $Z = \exp\left(-\theta W(T) - \frac{1}{2}\theta^2 T\right)$ where $\theta = (\alpha - r)/\sigma$ is the market price of risk, and

$$\widetilde{\mathbb{P}}(A) = \int_A Z\,\mathrm{d}\mathbb{P}, \quad A \in \mathcal{F}(T).$$

So with $\widetilde{W}(t) = W(t) + \theta t$

$$\mathrm{d}e^{-rt}S(t) = \sigma e^{-rt}S(t)\,\mathrm{d}\widetilde{W}(t)$$

and by Girsanov's theorem the discounted stock price is a martingale under $\widetilde{\mathbb{P}}$. Consider an agent who is short in a derivative security that pays $V(T)$ at time T. Suppose that he begins with initial capital $X(0)$ and at each time t, $0 \le t \le T$, holds $\Delta(t)$ shares of stock, investing or borrowing at the interest rate r as necessary to finance this. The differential of the agent's portfolio value $X(t)$ is

$$\mathrm{d}X(t) = \Delta(t)\,\mathrm{d}S(t) + r(X(t) - \Delta(t)S(t))\,\mathrm{d}t$$

and it follows that the differential of the discounted portfolio value is given by

$$\mathrm{d}e^{-rt}X(t) = \Delta(t)\sigma e^{-rt}S(t)\,\mathrm{d}\widetilde{W}(t).$$

So, the discounted portfolio value is a martingale. For a hedge of the derivative security we need $X(T) = V(T)$. The existence of a hedge follows from the martingale representation theorem. The value of the hedging portfolio is then defined as the price $V(t)$ of the derivative. The risk-neutral pricing formula is then

$$V(t) = X(t) = e^{rt} \cdot e^{-rt}X(t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}V(T) \mid \mathcal{F}(t)\right].$$

3. Consider a European call and a European put option both with strike K and exercise time T. Let $C(t)$ and $P(t)$ be the values at time $t \le T$ of the call and the put respectively. Consider two portfolios:

π_A : call option plus $Ke^{-r(T-t)}$ cash invested in a bank with constant interest rate r,

π_B : put option plus one unit of the asset with price process $\{S(t)\}$.

Derive the put-call parity

$$C(t) + Ke^{-r(T-t)} = P(t) + S(t).$$

Solution: The portfolios π_A and π_B have the same exercise value. By no-arbitrage the values at the times t, $0 \le t \le T$ are the same as well.

4. Let the stock price be modeled as a geometric Brownian motion

$$\mathrm{d}S(t) = \alpha S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}W(t), \quad 0 \le t \le T,$$

and let r denote the interest rate. The value at time t of a call option with strike K and expiry T is denoted by $C(t)$, and the value of a put option with the same strike and expiry is denoted by $P(t)$. Now consider a date t_0 between 0 and T, and consider a *chooser option*, which gives the right at time t_0 to choose to own either the call or the put.

(a) Show that at time t_0 the value of the chooser option is

$$C(t_0) + \left(e^{-r(T-t_0)}K - S(t_0)\right)^+.$$

Solution: the value at time t_0 is

$$\max(C(t_0), P(t_0)) = C(t_0) + \max(0, P(t_0) - C(t_0))$$

By the put-call parity we can write

$$\begin{aligned}\max(C(t_0), P(t_0)) &= C(t_0) + \max(0, Ke^{-r(T-t_0)} - S(t_0)) \\ &= C(t_0) + \left(e^{-r(T-t_0)}K - S(t_0)\right)^+.\end{aligned}$$

(b) Show that the value of the chooser option at time 0 is the sum of the value of a call expiring at time T with strike price K and the value of a put expiring at time t_0 with strike price $e^{-r(T-t_0)}K$.

Solution: The discounted value is a martingale with respect to the risk-neutral measure, so the value at time 0 is

$$\begin{aligned}&\widetilde{\mathbb{E}}\left[e^{-rt_0}\left(C(t_0) + \left(e^{-r(T-t_0)}K - S(t_0)\right)^+\right)\right] \\ &\qquad = C(0) + \widetilde{\mathbb{E}}\left[e^{-rt_0}\left(e^{-r(T-t_0)}K - S(t_0)\right)^+\right]\end{aligned}$$

which is the sum of the value of a call expiring at time T with strike price K and the value of a put expiring at time t_0 with strike price $e^{-r(T-t_0)}K$.

5. The price of a stock is modeled as a geometric Brownian motion

$$\mathrm{d}S(t) = \alpha S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}W(t), \quad S_0 = 5,$$

where $\alpha = 0.1$ and $\sigma = 0.2$. Consider a derivative with payoff

$$V(T) = S(T) - S(T/2)$$

at time $T = 4$. The interest rate of the bank account is $r = 0.02$.

(a) Calculate the probability that the payoff of the derivative is > 0.
Solution:

$$\begin{aligned} S(T) &= S(T/2)\exp\left\{\sigma(W(T)-W(T/2))+(\alpha-\frac{1}{2}\sigma^2)T/2\right\} \\ &= S(T/2)\exp\{0.2(W(4)-W(2))+0.16\} \end{aligned}$$

so

$$\{V(T)>0\}=\{0.2(W(4)-W(2))+0.16>0\}$$

and $\mathbb{P}(V(T)>0)=\mathbb{P}(U>-0.5657)$, where $U\sim N(0,1)$.

(b) Show that the value of the derivative at time $T/2$ is given by

$$V(T/2)=\left(1-e^{-rT/2}\right)S(T/2).$$

Solution:

$$\begin{aligned} e^{-rT/2}V(T/2) &= \widetilde{\mathbb{E}}\left[e^{-rT}V(T)\mid\mathcal{F}(T/2)\right] \\ &= \widetilde{\mathbb{E}}\left[e^{-rT}S(T)\mid\mathcal{F}(T/2)\right]-e^{-T}S(T/2) \\ &= (e^{-rT/2}-e^{-T})S(T/2) \end{aligned}$$

and it follows that

$$V(T/2)=\left(1-e^{-rT/2}\right)S(T/2).$$

(c) Find a hedge for a payoff $V(T/2)$ at time $T/2$.
Solution:
Set up a static hedge. Buy at time 0 $(e^{rT/2}-1)$ shares of the stock. The capital needed for buying the stock is equal to the price of the derivative.

(d) Find a hedge for the derivative with payoff $V(T)$ at time T.
Solution: We follow the argument in Shreve Section 5.3.2.

$$V(0)=\widetilde{\mathbb{E}}\left[e^{-rT}(S(T)-S(T/2))\right]=\left(1-e^{-rT/2}\right)S(0).$$

We have

$$\begin{aligned} e^{-rt}V(t) &= \widetilde{\mathbb{E}}\left[e^{-rT}(S(T)-S(T/2))\mid\mathcal{F}_t\right] \\ &= \begin{cases} \left(1-e^{-rT/2}\right)e^{-rt}S(t) & \text{if } t<T/2 \\ e^{-rt}S(t)-e^{-rT/2}S(T/2) & \text{if } t\geq T/2 \end{cases} \end{aligned}$$

Since

$$e^{-rt}S(t)=S(0)+\int_0^t\sigma e^{-ru}S(u)\,\mathrm{d}\widetilde{W}(u),$$

it follows that

$$e^{-rt}V(t) = V(0) + \int_0^t \widetilde{\Gamma}(u)\,\mathrm{d}\widetilde{W}(u)$$

where

$$\widetilde{\Gamma}(u) = \begin{cases} \sigma\left(1 - e^{-rT/2}\right)e^{-ru}S(u) & \text{if } u < T/2 \\ \sigma e^{-ru}S(u) & \text{if } u \geq T/2 \end{cases}$$

hence

$$\Delta(t) = \begin{cases} 1 - e^{-rT/2} & \text{if } t < T/2 \\ 1 & \text{if } t \geq T/2 \end{cases}$$