1

TW2530 13/04/15

a) $\dot{X}_1^* = 0 \Rightarrow X_2^* = 0$

$\dot{X}_2^* = 0 \Rightarrow -X_1^* - 2(X_1^*)^3 - (X_2^*)^3 = 0 \Rightarrow -X_1^* - 2(X_1^*)^3 = 0$

$\Rightarrow X_1^* = 0$ or $X_1^* = \pm i/\sqrt{2}$ (non-physical $x \in \mathbb{R}$)

Thus the only (physical) equilibrium point is $(0,0)$.

b) The linearized system is $\begin{cases} \dot{Z}_1 = Z_2 \\ \dot{Z}_2 = -Z_1 \end{cases}$

The eigenvalues are on the imaginary axis, therefore no conclusion can be drawn.

c) i $V(0,0) = 0$ &

ii $V(X_1, X_2) > 0$ for all $(X_1, X_2) \neq (0,0)$

iii $\dot{V} = -2X_2^4 \leq 0 \Rightarrow \dot{V}$ is negative semi definite

$\Rightarrow$ answer = no.

2

a) $Y(s) = \overbrace{\frac{h_1(s)}{1 - h_1(s)h_2(s)}}^{H(s)} U(s) = \frac{(s+3)(s-1)}{s^3 + 5s^2 + 12s + 15 - k} U(s)$

b) For $k = -3$ and $k = 33$, $H(s)$ has a two-dimensional realization. For $k = -3$:

$$H(s) = \frac{s-1}{s^2 + 2s + 6} \longrightarrow A = \begin{bmatrix} 0 & 1 \\ -6 & -2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; C = [-1 \ 1]$$

For $k = 33$:

$$H(s) = \frac{s+3}{s^2 + 6s + 18} \longrightarrow A = \begin{bmatrix} 0 & 1 \\ -18 & -6 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [3 \ 1]$$

c) For $-45 < k < 15$ the transfer function $H(s)$ is stable

3

a) The eigenvalues of A are $-\alpha-1, -\alpha-2, -2+\alpha$ and thus all have negative real part if and only if $-1<\alpha<2$.

b)

$$R = [B \;\; AB \;\; A^2B] = \left[\begin{array}{cc|c} 1 & -2 & 4+\alpha^2 \\ 0 & 0 & 0 \\ 0 & -\alpha & 4\alpha \end{array}\right]$$

Clearly, $\operatorname{rank}(R) \leq 2 < 3$, so that the system is not controllable for any $\alpha \in \mathbb{R}$.

c)

$$W = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 1 \\ \hline -2 & 3 & -\alpha \\ -\alpha & -\alpha-2 & -2 \\ \hline * & * & * \end{array}\right] \sim \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 1 \\ \hline 0 & 0 & -\alpha-3 \\ 0 & 0 & \alpha \\ \hline * & * & * \end{array}\right]$$

$\operatorname{rank}(W) = 3$, irrespective of $\alpha \in \mathbb{R}$. Hence, the system is observable for all $\alpha \in \mathbb{R}$.

d) Based on R, select

$$T = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right]. \text{ Then, } \tilde{A} = T^{-1}AT = \left[\begin{array}{cc|c} -2 & -1 & 3 \\ -1 & -2 & -1 \\ \hline 0 & 0 & -2 \end{array}\right]$$

$B = T^{-1}B = B$ and $\tilde{C} = CT = C$.

4/ Note that the eigenvalues are -4 (2x), 6 (1x).

a) For $\lambda = 6$, $\text{rank}(A - \lambda I \quad B) = 3 \Rightarrow$ hence, the system is stabilizable.

b) $F = [f_1 \; f_2 \; f_3]$. We must have

$$\det(\lambda I - (A + BF)) \equiv (\lambda + 4)^3.$$

It follows that $F = [-10 \quad 0 \quad 5]$ does the job.

c) For $\lambda = 6$

$$\text{rank}\begin{pmatrix} A - \lambda I \\ C \end{pmatrix} = \text{rank}\begin{bmatrix} -6 & 0 & -2 \\ 0 & -10 & 0 \\ -12 & 0 & -4 \\ \hline 0 & 1 & 1 \end{bmatrix} = 3 \Rightarrow \underline{\text{detect.}}$$

d) For $\lambda = -4$

$$\text{rank}\begin{pmatrix} A - \lambda I \\ C \end{pmatrix} = \text{rank}\begin{bmatrix} 4 & 0 & -2 \\ 0 & 0 & 0 \\ -12 & 0 & 6 \\ \hline 0 & 1 & 1 \end{bmatrix} = 2 < 3$$

$\Rightarrow$ not observable.

Hence, at least one eigenvalue of $A - KC$ must be equal to -4.

5/

a) False. Take $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then the pair (A^2, B) is stabilizable, but the pair (A, B) is not.

b) False. Take A the 2×2 zero matrix and C the 2×2 identity matrix. Then, (C,A) is observable, while A contains identical eigenvalues.

c) True. Let $h_1(s)$ be stable and $h_2(s)$ unstable. Then,

$$h(s) = h_1(s) + h_2(s)$$

$$= \frac{q_1(s)}{p_1(s)} + \frac{q_2(s)}{p_2(s)} = \frac{*}{p_1(s)\underbrace{p_2(s)}_{\text{poles in RHP}}}$$

$\Rightarrow$ unstable