Resit Logic (TW3520)
24-05-2016; 13:30–16:30.

The exam consists of multiple choice questions, 1–9, and open problems, 10–15.
Write down the answer of the multiple choice questions in a neat table on the exam paper.

Multiple choice questions

Each multiple choice question has exacly one correct answer.

(1) 1. Determine which of the following statement forms is logically implied by $(A \vee B)$:

(a) $(A \wedge \neg B)$.

(b) $(A \Leftrightarrow B)$.

(c) $(\neg B \Rightarrow A)$.

(d) $(\neg B \Rightarrow \neg A)$.

(1) 2. Consider the system L of propositional calculus. Determine which of the following statements is true:

(a) Some tautologies are not theorems of L.

(b) Some theorems of L are not tautologies.

(c) L is consistent if and only if there is a tautology which is not a theorem of L.

(d) All three statements (a), (b) and (c) are false.

(1) 3. Let $\mathcal{B}$ be a closed wf of a first order-theory T. Determine which of the following statements is true:

(a) Each model M of T is a model of $\mathcal{B}$.

(b) If $\mathcal{B}$ has a model M then M is also a model of T.

(c) There is a model M of T such that M is not a model of $\mathcal{B}$.

(d) No statement in (a), (b) and (c) is true.

(1) 4. Let t be the term $f_1^2(x_1, x_2)$. Determine in which of the following wfs t is free for x_1:

(a) $(\forall x_2)(A_1^1(x_1) \Rightarrow A_1^1(x_2))$.

(b) $(\forall x_2)(A_1^1(x_2) \Rightarrow A_1^1(x_1))$.

(c) $(\forall x_2)A_1^1(x_2) \Rightarrow A_1^1(x_1)$.

(d) $(\forall x_2)A_1^1(x_1) \Rightarrow A_1^1(x_2)$.

(1) 5. Let K be a first-order theory in the language $\mathcal{L}$. Determine which of the following is true:

(a) If K has a denumerable model then K has a finite model.

(b) If $\mathcal{L}$ is denumerable then every model of K is denumerable.

(c) If there is a finite interpretation of $\mathcal{L}$ then there is a finite model of K.

(d) If there is a finite model of K then there is a finite interpretation of $\mathcal{L}$.

See also the next page.

(1) 6. Let $\mathcal{F}$ be an ultrafilter on ω . Determine which of the following is true:

(a) There is a finite A such that $A \in \mathcal{F}$.

(b) There is a finite A such that $\omega \setminus A \in \mathcal{F}$.

(c) For each A, either A or $\omega \setminus A$ is an element of $\mathcal{F}$.

(d) No statement in (a), (b) and (c) is true.

(1) 7. Let $\mathcal{F}$ be a nonprincipal ultrafilter on ω and $\mathcal{R}^* = \mathcal{R}^\omega/\mathcal{F}$ a nonstandard model of analysis. Determine which of the following is true:

(a) There is an infinitely large $x \in (1, 2)^*$.

(b) There is an infinitely small $x \in (1, 2)^*$.

(c) If $A \subseteq (1, 2)$ then $A^* \neq A$.

(d) There is an $A \subseteq (1, 2)$ such that $A^* = A$.

(1) 8. Let M and N be models of a first-order theory. Determine which of the following is true:

(a) If M is a substructure of N and M and N are isomorphic, then M is an elementary substructure of N.

(b) If M and N are elementarily equivalent, then M and N are isomorphic.

(c) If M is an elementary substructure of N, then M and N are isomorphic.

(d) No statement in (a), (b) and (c) is true.

(1) 9. Let S be the formal number theory. One of the consequences of the first Gödel's incompleteness theorem is:

(a) If S is consistent then it is complete.

(b) If S is complete then it is inconsistent.

(c) S is inconsistent.

(d) None of the above is a consequences of the first Gödel's incompleteness theorem.

See also the next page.

Open problems

The answer to any open problem must be given with all details.

(2) 10. By introducing appropriate notation, determine if the following set of statements is consistent.

Either the passport was forged or, if Peter left the country, he had enough money. If the passport was forged, then Peter did not leave the country. If Peter had enough money, then he left the country.

(3) 11. Let $\mathcal{B}$ be the following wf of the propositional calculus L:

$$((A \Rightarrow B) \wedge B) \Rightarrow A.$$

Let L^+ be the formal theory obtained from L by adding as new axioms all wfs obtainable from $\mathcal{B}$ by substituting arbitrary statement forms for the statement letters in $\mathcal{B}$, with the same form being substituted for all occurrences of a statement letter. Show that L^+ is inconsistent.

(3) 12. Prove the following:

$$\vdash (\exists x)(\mathcal{B}(x) \Rightarrow \mathcal{C}(x)) \Rightarrow ((\forall x)\mathcal{B}(x) \Rightarrow (\exists x)\mathcal{C}(x)).$$

Hint: Use rule C.

(1) 13. a. State the compactness theorem for first-order theories.

(3) b. Prove the compactness theorem for first-order theories.

(2) 14. Prove that a filter $\mathcal{F}$ on ω is a principal filter if and only if the intersection of all sets in $\mathcal{F}$ is an element of $\mathcal{F}$.

15. Let $\mathcal{F}$ be a nonprincipal ultrafilter on ω. Consider the ultrapower $\mathcal{R}^* = \mathcal{R}^\omega/\mathcal{F}$

(1) a. Let f be a real-valued function on a set B of real numbers. Let $c \in B$. Let B^* be the subset of $\mathbb{R}^*$ corresponding to B, and let f^* be the function corresponding to f. Give the nonstandard definition of "f is continuous at c".

(3) b. By using the nonstandard definition of continuity prove or disprove that the following function is continuous at 1:

$$f(x) = \begin{cases} \dfrac{x^2 - 2x + 1}{x^2 - 1} & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$$

See also the next page.

Axiom System L for the Propositional Calculus:
If $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are wfs of L then the following are axioms of L:

(A1) $(\mathcal{B} \Rightarrow (\mathcal{C} \Rightarrow \mathcal{B}))$

(A2) $((\mathcal{B} \Rightarrow (\mathcal{C} \Rightarrow \mathcal{D})) \Rightarrow ((\mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow (\mathcal{B} \Rightarrow \mathcal{D})))$

(A3) $(((\neg\mathcal{C}) \Rightarrow (\neg\mathcal{B})) \Rightarrow (((\neg\mathcal{C}) \Rightarrow \mathcal{B}) \Rightarrow \mathcal{C}))$

The only rule of inference of L is modus ponens (abbreviation MP):
$\mathcal{C}$ is a direct consequence of $\mathcal{B}$ and $(\mathcal{B} \Rightarrow \mathcal{C})$

Axiom System for a First-Order Theory K:
Let $\mathcal{L}$ be a first-order language. If $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are wfs in $\mathcal{L}$, then the following are logical axioms of K:

(A1) $\mathcal{B} \Rightarrow (\mathcal{C} \Rightarrow \mathcal{B})$

(A2) $((\mathcal{B} \Rightarrow (\mathcal{C} \Rightarrow \mathcal{D})) \Rightarrow ((\mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow (\mathcal{B} \Rightarrow \mathcal{D}))$

(A3) $(\neg\mathcal{C} \Rightarrow \neg\mathcal{B}) \Rightarrow ((\neg\mathcal{C} \Rightarrow \mathcal{B}) \Rightarrow \mathcal{C})$

(A4) $(\forall x_i)\mathcal{B}(x_i) \Rightarrow \mathcal{B}(t)$ if $\mathcal{B}(x_i)$ is a of $\mathcal{L}$ and t is a term of $\mathcal{L}$ that is free for x_i in $\mathcal{B}(x_i)$

(A5) $(\forall x_i)(\mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow (\mathcal{B} \Rightarrow (\forall x_i)\mathcal{C})$ if $\mathcal{B}$ contains no free occurences of x_i

Proper axioms depend on the first-order theory.
The following wfs are axioms of a first-order theory with equality:

(A6) $(\forall x_1)x_1 = x_1$

(A7) $x = y \Rightarrow (\mathcal{B}(x, x) \Rightarrow \mathcal{B}(x, y))$

The rules of inference of any first-order theory are:

Modus ponens (abbreviation MP): $\mathcal{C}$ follows from $\mathcal{B}$ and $\mathcal{B} \Rightarrow \mathcal{C}$

Generalization (abbreviation Gen): $(\forall x_i)\mathcal{B}$ follows from $\mathcal{B}$

The mark for each question is shown in the margin; the final grade is calculated using the formula

$$\text{Grade} = \frac{\text{Total} + 3}{3}$$

and rounded in the usual manner.

THE END