Faculty of Electrical Engineering, Mathematics and Computer Science
Numerical Methods II, TW3530, BSc Mathematics
EXAM APRIL 18, 2017, 9:00 – 12:00, EWI-Lecture hall G

1 Given the following boundary value problem on $\Omega = (0,1) \times (0,1)$ with boundary $\partial\Omega$:

$$(P_1): \begin{cases} -\nabla \cdot (D(x,y)\nabla u) = f(x,y), & \text{in } \Omega, \\ D(x,y)\dfrac{\partial u}{\partial n} + u = g(x,y), & \text{on } \partial\Omega, \end{cases}$$

where $f(x,y)$ and $g(x,y)$ are given functions. Further, $D(x,y)$ is a given function for which $\boxed{D(x,y) > 0}$ in Ω. We use the finite volume method to solve the above problem. We construct the grid such that the gridsize is constant and such that we have unknowns on the boundary segments, including on the vertices of the boundary. We use n unknowns per dimension, hence the total number of unknowns $N = n^2$. Further, we use a horizontal numbering arrangement, that is, the first n unknowns are located on boundary segment $\{y = 0\}$. In this assignment, we derive the system of linear equations

$$A\underline{v} = \underline{f}. \tag{1}$$

a Derive the finite volume method for an *internal* gridnode (so a gridnode that is not positioned on $\partial\Omega$). Use i as the index of the node. (2pt.)

b Give the entries of the i-th row of the discretisation matrix A (see equation (1)). (2pt.)

c Derive the finite volume method for a point on the boundary. You are allowed to choose any point on the boundary and you only have to do this for one chosen point on $\partial\Omega$. Use k as the index of the point that you consider. (2pt.)

d Give the entries of the k-th row of the discretisation matrix A (see equation (1)). (2pt.)

e Next, we consider the eigenvalues of A by the use of Gershgorin's Theorem, which reads as

Theorem: Let A be an $N \times N$–matrix, let λ be an eigenvalue of A, then the eigenvalues of A are located in the complex plane within the union of the disks defined by

$$|\lambda - a_{ii}| \leq \sum_{j=1_{j\neq i}}^{N} |a_{ij}|. \tag{2}$$

Use Gerschgorin's Theorem to derive lower and upper bounds for the eigenvalues of A (use symmetry of A and it is sufficient to base your proof on your earlier results in b and d of this assignment). (2pt.)

2 Given the following initial boundary value problem for $u = u(t,(x,y))$, on $\Omega = (0,1) \times (0,1)$ with boundary $\partial\Omega$ and for $t > 0$:

$$(P_2): \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = 0, & \text{for } t > 0,\ (x,y) \in \Omega, \\ u = 0, & \text{for } t > 0,\ (x,y) \in \partial\Omega, \\ u(0,(x,y)) = u_0(x,y), \underbrace{\dfrac{\partial u}{\partial t} = v_0(x,y)}_{\text{tijds afgl.}}, & \text{for } (x,y) \in \Omega. \end{cases}$$

a We consider classical solutions that have at least continuous second-order partial derivatives with respect to time and a continuous second-order partial derivative with respect to time.

We demonstrate conservation of energy, that is, show that

$\frac{\partial u}{\partial t}(0,x,y) = V_0(x,y).$

$$\int_\Omega (\frac{\partial u}{\partial t})^2 + ||\nabla u||^2 d\Omega = C, \tag{3}$$

where C is a constant, further, determine the value of C in terms of the initial conditions. *Hints: (1) Multiply the PDE by $\frac{\partial u}{\partial t}$, and integrate over Ω, (2) use integration by parts, (3) use the Chain Rule for differentiation, (4) interchange differentiation with respect to time and position and (5) use the boundary condition.* (3pt.)

b Use the finite difference method with horizontal numbering and with n unknowns per coordinate direction to construct a numerical approximation of the form

$$\underline{v}'' + A\underline{v} = \underline{f}, \qquad (4)$$

where $\underline{v} = [v_1 \dots v_N]^T$, with $N = n^2$ and where $v_i(t)$ represents the numerical approximation of the solution $u(t,(x_i,y_i))$. Describe the matrice A and the vector $\underline{f}$. Give the initial condition for $\underline{v}$. (3pt.)

c Describe how you would solve system (4) by the use of reduction of order (for the time-derivative) and the backward Euler time-integration method (Euler Implicit). (2pt.)

d Do you expect that your numerical solution will be subject to dissipation ('loss of signal') or to amplification ('growth of signal') upon using the backward Euler method? Motivate your answer (Hint: You may assume that the temporal behaviour of the numerical solution is identical to the temporal behaviour of the numerical solution to $y' = i\mu y$, where i represents the imaginary unit). (2pt.)

$$\text{Exam grade} = \frac{\text{Total number of points}}{2}.$$

euler backward $w_{n+1} = w_n + \Delta t\, f(t_{n+1}, w_{n+1})$

$$\begin{bmatrix} \varnothing & -M^{-1}S \\ I & \varnothing \end{bmatrix}$$

$y' = i\mu y$

$$\boxed{Q(\lambda\Delta t) = \frac{1}{1-\lambda\Delta t}}$$

$\int f\,dg = fg - \int g\,df$

$\int uu'\,dt$

$$\int \frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\,d\Omega = \left[\frac{\partial u}{\partial t}\, u\right] - \int u \frac{\partial^2 u}{\partial t \partial x}\,d\Omega$$