Exam Discrete Optimization (WI4227)

21 january 2014, 14.00 – 17.00 (3 hours).

This exam consists of 5 problems worth 90 pts in total. Please write your name on every sheet (including this one).
During the exam, books, written notes, graphical calculators and mobile phones are *not allowed*.
Good luck!

Name: ______ **Student number:** ______

[18pts] **1.** The following questions are worth 2pts each. In each question check the box(es) of the correct answer(s).

(a) Which of the following statements are true?

- ☐ For any matrix A, the collection of linearly independent sets of columns form the independent sets of a matroid.
- ☐ For any bipartite graph G, the collection of stable sets form the independent sets of a matroid.
- ☐ For any graph G, the collection
  $$\{I \subseteq V \mid \text{there is a matching } M \text{ in } G \text{ that covers all nodes in } I\}$$
  is the collection of independent sets of a matroid.

(b) Which of the following statements are true?

- ☐ If B and B' are bases of a matroid M, then for every $x \in B \setminus B'$ and every $y \in B' \setminus B$ the set $(B \cup \{y\}) \setminus \{x\}$ is a basis of M.
- ☐ If r is the rank function of a matroid, then $r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$ holds for all subsets A and B of the ground set.
- ☐ If r is the rank function of a matroid on the ground set S, then the set
  $$P = \{x \in \mathbb{R}^S \mid x \geq \mathbf{0},\ x(U) \leq r(U) \text{ for every } U \subseteq S\}$$
  is an integral polytope.

(c) Given an undirected graph $G = (V, E)$ two nodes $s, t \in V$ and a cost function $c : E \to \mathbb{Z}$, a minimum cost s–t path can be found in polynomial time

- ☐ using the algorithm of Bellmann-Ford, after replacing each edge by two opposite arcs of the same cost as the edge, provided that G has no negative cost cycles.
- ☐ using the algorithm of Dijkstra, provided that all costs are nonnegative.
- ☐ by solving a corresponding matching problem using the Blossom algorithm, provided that G has no negative cost cycles.

(d) Consider the Traveling Salesman Problem. Which of the following statements are true?

- ☐ For metric TSP, the heuristic of Christofides always gives a solution that is within a factor 1.5 of optimum.
- ☐ The subtour bound can be computed in polynomial time using the ellipsoid method.
- ☐ For metric TSP, the nearest neighbour heuristic always gives a solution that is within a factor 10^{52} of optimum.

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(e) Which statements about the Blossom algorithm for minimum weight perfect matching are true?

- ☐ At any stage of the algorithm, any perfect matching in G that uses only tight edges is automatically a minimum weight perfect matching.
- ☐ The algorithm cannot be applied to bipartite graphs, because they have no blossoms.
- ☐ When contracting a blossom, it becomes an *even* (pseudo) node.

(f) Let A be a square matrix. Which of the following conditions imply that A is Totally Unimodular?

- ☐ The determinant of A equals -1, 0 or 1.
- ☐ Every column of A contains exactly two nonzero entries: a 1 and a -1.
- ☐ For every integral b, the polytope $\{x \mid Ax \leq b, x \geq \mathbf{0}\}$ is integral (or empty).

(g) Let $P := \{x \mid Ax \leq b\} \subseteq \mathbb{R}^n$ be a polyhedron, where A and b are integral. Which statements are true?

- ☐ If P is pointed, then P is a polytope.
- ☐ P is a polytope if and only if A has rank n.
- ☐ If P is polytope, then any inequality $c^\mathsf{T} x \leq d$ valid for the integer hull of P has a cutting plane proof starting from the system $Ax \leq b$.

(h) Given a capacitated network, a flow from s to t under the capacity, and an s–t cut $\delta^{\text{out}}(U)$. If the value of the flow equals the capacity of the cut, then what can we conclude?

- ☐ On each arc leaving U, the amount of flow equals the capacity of the arc.
- ☐ On each arc entering U, the amount of flow equals zero.
- ☐ The amount of flow leaving U minus the amount entering U equals the flow value.

(i) A matching M in a bipartite graph has maximum size if and only if

- ☐ there is a vertex cover U of size $|U| = |M|$.
- ☐ no edge can be added to M to create a larger matching.
- ☐ there is no M-augmenting path.

Hand in this sheet!

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[18pts] **2.** Let $P \subseteq \mathbb{R}^3$ be defined by

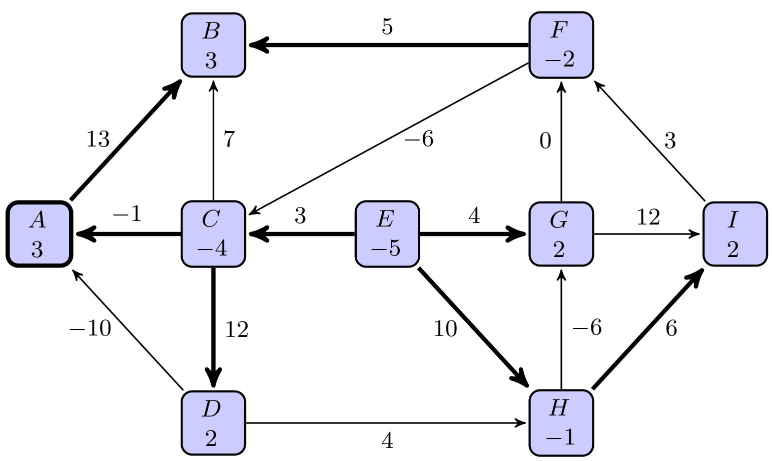
$$x \geq \mathbf{0}, \quad x_1 + 2x_2 + 3x_3 \leq 6. \tag{1}$$

[3pts] (a) Give the definition of Totally Dual Integral (TDI).

[5pts] (b) Show that system (1) is not TDI.

[10pts] (c) Give a minimal TDI system $Ax \leq b$ describing P with A and b integral. Here *minimal* means that none of the inequalities can be left out.

[18pts] **3.** Consider the network in the figure below, which consists of a directed graph, a demand function b on the nodes, and costs c on the arcs. There are no upper bounds: $u(a) = \infty$ for every arc a. Use the network simplex method to solve the minimum cost flow problem, starting from the given tree solution (thick arcs). The vertex A is the root. In each iteration, give the tree, the associated flow, the vector y (the cost of the paths in the tree from the root to the nodes) and the cost of the flow.



[18pts] **4.** In a 3×3 table, we want to choose at least 3 cells, but in such a way that from each row, from each column, and from each of the two diagonals at most one cell is chosen. We first consider an LP relaxation.

Let $P \subseteq \mathbb{R}^{3\times 3}$ be the polytope defined by the system (2)–(5):

$$x \geq \mathbf{0}, \tag{2}$$

$$\sum_{i,j=1}^{3} x_{ij} \geq 3, \tag{3}$$

$$x_{1i} + x_{2i} + x_{3i} \leq 1 \quad \text{and} \quad x_{i1} + x_{i2} + x_{i3} \leq 1 \qquad (\ i = 1, 2, 3\ ), \tag{4}$$

$$x_{11} + x_{22} + x_{33} \leq 1 \quad \text{and} \quad x_{13} + x_{22} + x_{31} \leq 1. \tag{5}$$

[3pts] (a) Let

$$z := \begin{bmatrix} \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}.$$

Determine the inequalities that are tight at z and show that z is a vertex of P.

[5pts] (b) Derive from the given system the cutting plane

$$x_{22} \leq 0. \tag{6}$$

[5pts] (c) Show that inequalities (5) are implied by inequalities (2)–(4) and (6).
Hint. First show that $x_{12} + x_{32} + x_{21} + x_{23} \geq 2$.

[5pts] (d) Show that the system of inequalities (2)–(4) and (6) can be written as $Ax \leq b$ with A a totally unimodular matrix and b integral. Conclude that this system determines the integer hull of P.

[18pts] **5.** In the figure below, you see a network with costs associated to the edges. The problem is to find a walk from A to H of minimum total cost that traverses all edges *at least once*.

[6pts] (a) Formulate this as a minimum cost T-join problem for some subset T of the nodes. Explain why solving the T-join problem solves the original problem.

[6pts] (b) Explain how this T-join problem can be reduced to a weighted matching problem. Also give the corresponding weight function.

[6pts] (c) Solve the matching problem and give a minimum cost walk from A to H traversing every edge of the graph at least once.