EXAM DISCRETE OPTIMIZATION (WI4227) SOLUTIONS

21 january 2014, 14.00 – 17.00 (3 hours).

[18pts] **1.** The following questions are worth 2pts each. In each question check the box(es) of the correct answer(s).

(a) Which of the following statements are true?

- ■ For any matrix A, the collection of linearly independent sets of columns form the independent sets of a matroid.
- ☐ For any bipartite graph G, the collection of stable sets form the independent sets of a matroid.
- ■ For any graph G, the collection
  $\{I \subseteq V \mid \text{there is a matching } M \text{ in } G \text{ that covers all nodes in } I\}$
  is the collection of independent sets of a matroid.

(b) Which of the following statements are true?

- ☐ If B and B' are bases of a matroid M, then for every $x \in B \setminus B'$ and every $y \in B' \setminus B$ the set $(B \cup \{y\}) \setminus \{x\}$ is a basis of M.
- ■ If r is the rank function of a matroid, then $r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$ holds for all subsets A and B of the ground set.
- ■ If r is the rank function of a matroid on the ground set S, then the set
  $P = \{x \in \mathbb{R}^S \mid x \geq \mathbf{0},\ x(U) \leq r(U) \text{ for every } U \subseteq S\}$
  is an integral polytope.

(c) Given an undirected graph $G = (V, E)$ two nodes $s, t \in V$ and a cost function $c : E \to \mathbb{Z}$, a minimum cost s–t path can be found in polynomial time

- ☐ using the algorithm of Bellmann-Ford, after replacing each edge by two opposite arcs of the same cost as the edge, provided that G has no negative cost cycles.
- ■ using the algorithm of Dijkstra, provided that all costs are nonnegative.
- ■ by solving a corresponding matching problem using the Blossom algorithm, provided that G has no negative cost cycles.

(d) Consider the Traveling Salesman Problem. Which of the following statements are true?

- ■ For metric TSP, the heuristic of Christofides always gives a solution that is within a factor 1.5 of optimum.
- ■ The subtour bound can be computed in polynomial time using the ellipsoid method.
- ☐ For metric TSP, the nearest neighbour heuristic always gives a solution that is within a factor 10^{52} of optimum.

(e) Which statements about the Blossom algorithm for minimum weight perfect matching are true?

- ■ At any stage of the algorithm, any perfect matching in G that uses only tight edges is automatically a minimum weight perfect matching.
- ☐ The algorithm cannot be applied to bipartite graphs, because they have no blossoms.
- ■ When contracting a blossom, it becomes an *even* (pseudo) node.

(f) Let A be a square matrix. Which of the following conditions imply that A is Totally Unimodular?

- ☐ The determinant of A equals -1, 0 or 1.
- ■ Every column of A contains exactly two nonzero entries: a 1 and a -1.
- ■ For every integral b, the polytope $\{x \mid Ax \leq b, x \geq \mathbf{0}\}$ is integral (or empty).
  We tacitly assume that A is integral, hence it follows from the theorem of Hoffman-Kruskal that the condition is sufficient. If the matrix is not integral the condition is not sufficient. Since this assumption was not clear, both True and False are accepted as correct answers.

(g) Let $P := \{x \mid Ax \leq b\} \subseteq \mathbb{R}^n$ be a polyhedron, where A and b are integral. Which statements are true?

- ☐ If P is pointed, then P is a polytope.
- ☐ P is a polytope if and only if A has rank n.
- ■ If P is polytope, then any inequality $c^\mathsf{T} x \leq d$ valid for the integer hull of P has a cutting plane proof starting from the system $Ax \leq b$.

(h) Given a capacitated network, a flow from s to t under the capacity, and an s–t cut $\delta^{\text{out}}(U)$. If the value of the flow equals the capacity of the cut, then what can we conclude?

- ■ On each arc leaving U, the amount of flow equals the capacity of the arc.
- ■ On each arc entering U, the amount of flow equals zero.
- ■ The amount of flow leaving U minus the amount entering U equals the flow value.

(i) A matching M in a bipartite graph has maximum size if and only if

- ■ there is a vertex cover U of size $|U| = |M|$.
- ☐ no edge can be added to M to create a larger matching.
- ■ there is no M-augmenting path.

[18pts] **2.** Let $P \subseteq \mathbb{R}^3$ be defined by

$$x \geq \mathbf{0}, \quad x_1 + 2x_2 + 3x_3 \leq 6. \tag{1}$$

[3pts] (a) Give the definition of Totally Dual Integral (TDI).

Solution. A rational system $Ax \leq b$ is *totally dual integral* if for every integral w the minimum in

$$\min y^\mathsf{T} b \text{ subject to } y^\mathsf{T} A = w^\mathsf{T}, y \geq \mathbf{0}$$

has an integral optimal solution, provided that the minimum exists (is attained). ■

[5pts] (b) Show that system (1) is not TDI.

Solution. Take $w^\mathsf{T} = (0, 0, 1)$. The dual program reads

$$\min 6y_0 \text{ subject to } y_0(1,2,3) + y_1(-1,0,0) + y_2(0,-1,0) + y_3(0,0,-1) = (0,0,1)$$

Clearly, the minimum equals 2, and the only optimal solution is given by $(y_0, y_1, y_2, y_3) = (\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, 0)$, which is not integral. ■

[10pts] (c) Give a minimal TDI system $Ax \leq b$ describing P with A and b integral. Here *minimal* means that none of the inequalities can be left out.

Solution. Let $P = \{x \in \mathbb{R}^3 \mid Ax \leq b\}$, where the system $Ax \leq b$ is given by

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &\leq 6, \\ -x_1 &\leq 0, \\ -x_2 &\leq 0, \\ -x_3 &\leq 0. \end{aligned}$$

The four vertices of P are $(0,0,2)$, $(0,3,0)$, $(6,0,0)$ and $(0,0,0)$. It suffices to find for each vertex a (minimal) Hilbert basis of the cone spanned by the rows of A that are tight at that vertex.

$(0,0,2)$ Rows 1,2,3 are tight. If $\lambda_1(1,2,3) + \lambda_2(-1,0,0) + \lambda_3(0,-1,0)$ is an integral vector, with $0 \leq \lambda_1, \lambda_2, \lambda_3 < 1$, then we must have $\lambda_3 \in \{0, \frac{1}{3}, \frac{2}{3}\}$ and for each case the values of λ_1 and λ_2 are uniquely determined. We obtain $\lambda = (0,0,0)$, $\lambda = (\frac{1}{3}, \frac{2}{3}, \frac{1}{3})$ and $\lambda = (\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$ respectively, with corresponding inequalities $(0,0,0)x \leq 0$, $(0,0,1)x \leq 2$, $(0,1,2)x \leq 4$. The first inequality, $(0,0,0)x \leq 0$, is to be ommitted.

$(0,3,0)$ Rows 1,2,4 are tight. For $0 \le \lambda_1, \lambda_2, \lambda_3 < 1$, the only possible nonzero integral vector of the form $\lambda_1(1,2,3) + \lambda_2(-1,0,0) + \lambda_3(0,0,-1)$ is $(0,1,1)$ with $\lambda = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, so we get $(0,1,1)x \le 3$.

$(6,0,0)$ Rows 1,3,4 are tight. There is no nonzero integer vector of the form $\lambda_1(1,2,3) + \lambda_2(0,-1,0) + \lambda_3(0,0,-1)$ where $0 \le \lambda_1, \lambda_2, \lambda_3 < 1$.

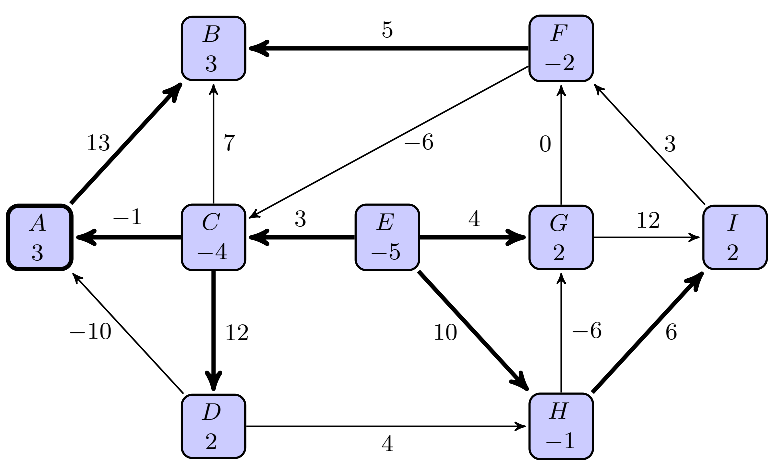
$(0,0,0)$ Rows 2,3,4 are tight. They are already a Hilbert base of the cone they span.

So in total we obtain the TDI system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &\le 6 \\ -x_1 &\le 0, \\ -x_2 &\le 0, \\ -x_3 &\le 0, \\ x_3 &\le 2, \\ x_2 + 2x_3 &\le 4, \\ x_2 + x_3 &\le 3. \end{aligned}$$

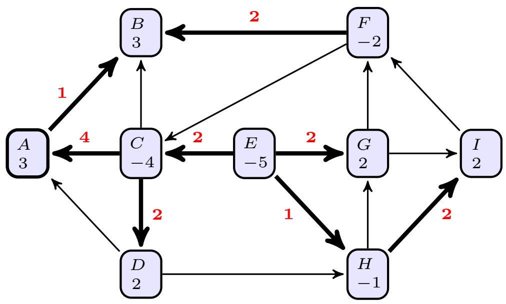
■

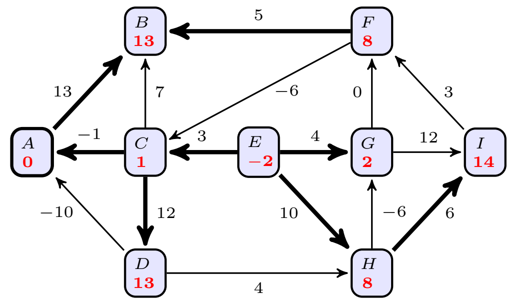
[18pts] **3.** Consider the network in the figure below, which consists of a directed graph, a demand function b on the nodes, and costs c on the arcs. There are no upper bounds: $u(a) = \infty$ for every arc a. Use the network simplex method to solve the minimum cost flow problem, starting from the given tree solution (thick arcs). The vertex A is the root. In each iteration, give the tree, the associated flow, the vector y (the cost of the paths in the tree from the root to the nodes) and the cost of the flow.



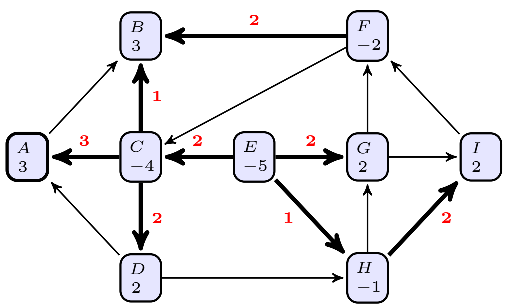
Solution. Below, we show the different iterations, starting with the initial basic solution. In each iteration, we denote the edges in the tree solution in bold. In the lefthand figures, we show the given demands of the nodes (black) and the flow values (in red). In the righthand figures, we show the given arc costs (black) and the values of y (in red).

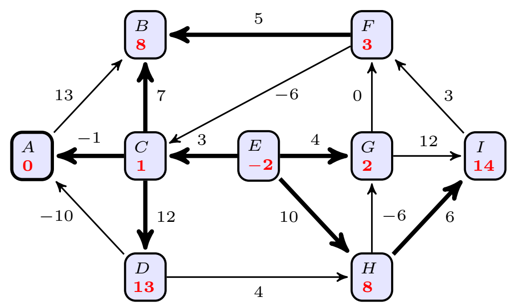
Iteration 0



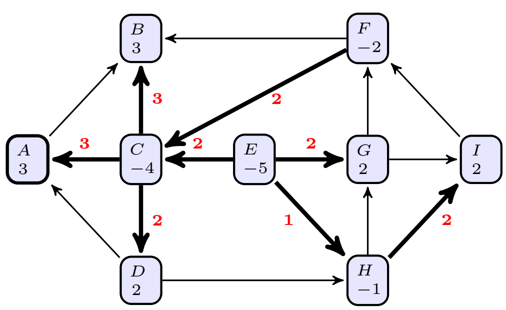


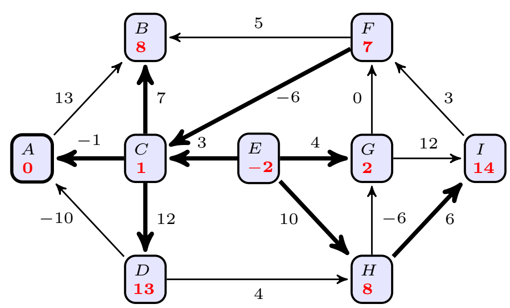
Iteration 1 Since $y_B - y_C = 12 > 7$ we add arc (B, C), creating the cycle (C, B, A, C). We increase the flow by 1 and the reverse arc (A, B) drops out of the tree solution. We obtain:



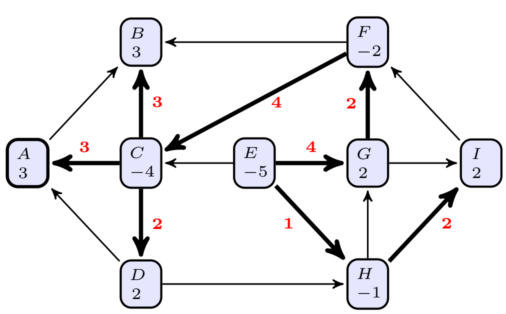


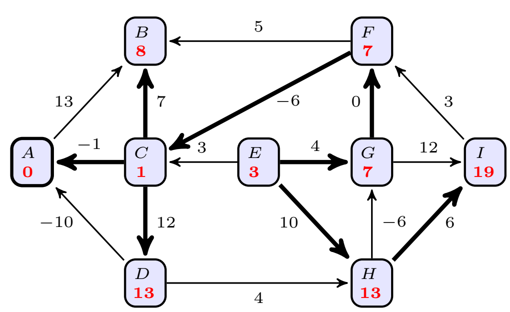
Iteration 2 Since $y_C - y_F = -2 > -6$ we add arc (F, C), creating the cycle (F, C, B, F). We increase the flow by 2 and the reverse arc (F, B) drops out of the tree solution. We obtain:





Iteration 3 Since $y_F - y_G = 5 > 0$ we add arc (G, F), creating the cycle $(G, F, , C, E, G)$. We increase the flow by 2 and the reverse arc (E, C) drops out of the tree solution. We obtain:





Iteration 4 There are no arcs (a, b) for which $y_a - y_b > c_{ab}$, hence we have an optimal solution. The value equals $3 \times 7 + 3 \times -1 + 2 \times 12 + 4 \times -6 + 2 \times 0 + 4 \times 4 + 1 \times 10 + 2 \times 6 = 56$. ∎

[18pts] **4.** In a 3×3 table, we want to choose at least 3 cells, but in such a way that from each row, from each column, and from each of the two diagonals at most one cell is chosen. We first consider an LP relaxation.

Let $P \subseteq \mathbb{R}^{3\times 3}$ be the polytope defined by the system (2)–(5):

$$x \geq \mathbf{0}, \tag{2}$$

$$\sum_{i,j=1}^{3} x_{ij} \geq 3, \tag{3}$$

$$x_{1i} + x_{2i} + x_{3i} \leq 1 \quad \text{and} \quad x_{i1} + x_{i2} + x_{i3} \leq 1 \qquad (\ i = 1, 2, 3\ ), \tag{4}$$

$$x_{11} + x_{22} + x_{33} \leq 1 \quad \text{and} \quad x_{13} + x_{22} + x_{31} \leq 1. \tag{5}$$

[3pts] (a) Let

$$z := \begin{bmatrix} \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}.$$

Determine the inequalities that are tight at z and show that z is a vertex of P.

Solution. The following inequalities are tight. From (2) we have tight inequalities $x_{12} \geq 0$, $x_{23} \geq 0$, $x_{31} \geq 0$. Inequality (3) is tight. From (4) and (5) all inequalities are tight. To show that z is a vertex, it suffices to show that the coefficient matrix corresponding to the tight inequalities, namely

$$\begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix},$$

has rank 9. Removing the first three rows and columns 2,6,7 corresponding to the -1 entries, and also removing the fourth row (which is the sum of rows 5,6,7) it suffices to show that the resulting matrix (on left) has rank 6. Bringing this matrix into echolon form, we find the matrix on the right, which indeed has rank 6.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

■

[5pts] (b) Derive from the given system the cutting plane

$$x_{22} \leq 0. \tag{6}$$

Solution. Adding the inequalities

$$\begin{aligned} -\sum_{i,j=1}^{3} x_{ij} &\leq -3 \\ x_{12}+x_{22}+x_{32} &\leq 1 \\ x_{21}+x_{22}+x_{23} &\leq 1 \\ x_{11}+x_{22}+x_{33} &\leq 1 \\ x_{13}+x_{22}+x_{31} &\leq 1 \end{aligned}$$

each with multiplicity $\frac{1}{3}$, we obtain $x_{22} \leq \frac{1}{3}$. Rounding down the righthand-side, we find $x_{22} \leq 0$. ■

[5pts] (c) Show that inequalities (5) are implied by inequalities (2)–(4) and (6).

Solution. Adding $-\sum_{i,j=1}^{3} x_{ij} \leq -3$, $x_{11}+x_{12}+x_{13} \leq 1$, $x_{31}+x_{32}+x_{33} \leq 1$, $x_{11}+x_{21}+x_{31} \leq 1$, $x_{13}+x_{23}+x_{33} \leq 1$, and $x_{22} \leq 0$ (multiplicity 3), we obtain $(x_{11}+x_{22}+x_{33})+(x_{31}+x_{22}+x_{33}) \leq 1$.
Together with the nonnegativity of $x_{11}, x_{22}, x_{13}, x_{31}, x_{33}$, this implies that $x_{11}+x_{22}+x_{33} \leq 1$ and $x_{31}+x_{22}+x_{12} \leq 1$. ■

[5pts] (d) Show that the system of inequalities (2)–(4) and (6) can be written as $Ax \leq b$ with A a totally unimodular matrix and b integral. Conclude that this system determines the integer hull of P.

Solution. If matrix A is TU, then the matrix obtained from A by adding a row containing a single nonzero entry, which equals 1, is also TU. Hence it suffices to show that the coefficient matrix corresponding to the constraints (3) and (4) is TU. This follows since $\{\{1,\ldots,9\},\{1,2,3\},\{4,5,6\},\{7,8,9\}\}$ and $\{\{1,4,7\},\{2,5,8\}.\{3,6,9\}\}$ are two nested families and hence the matrix is (up to resigning the rows) the incidence marix of the union of two nested families.
For a TU matrix A and an integral vector b, the polyhedron $\{x \mid Ax \leq b\}$ is integral, completing the proof. ■

[18pts] **5.** In the figure below, you see a network with costs associated to the edges. The problem is to find a walk from A to H of minimum total cost that traverses all edges *at least once*.

[6pts] (a) Formulate this as a minimum cost T-join problem for some subset T of the nodes. Explain why solving the T-join problem solves the original problem.

Solution. If nodes A and H had odd degree, and all other vertices had even degree, there would be a walk from A to H traversing every edge exactly once (Eulerian walk). Hence we want to double a subset F of the edges to achieve these degree parities, minimizing the total cost of the edges in F. Since the nodes of odd degree are A, B, E, J, we want to change the parity of the set of nodes $\{B, E, J, H\} = \{A, B, E, J\}\Delta\{A, H\}$. That is, we want to double the edges of a minimum cost T-join, where $T = \{B, E, J, H\}$, and then construct an Eulerian walk from A to H in the resulting graph. ■

[6pts] (b) Explain how this T-join problem can be reduced to a weighted matching problem. Also give the corresponding weight function.

Solution. We make the complete graph H on the nodes of T, and take the weight $w(\{s,t\})$ of each edge st to be the minimum cost of an s–t path in the original graph G. Any perfect matching M in H gives rise to a set of $|M|/2$ paths in G linking up the

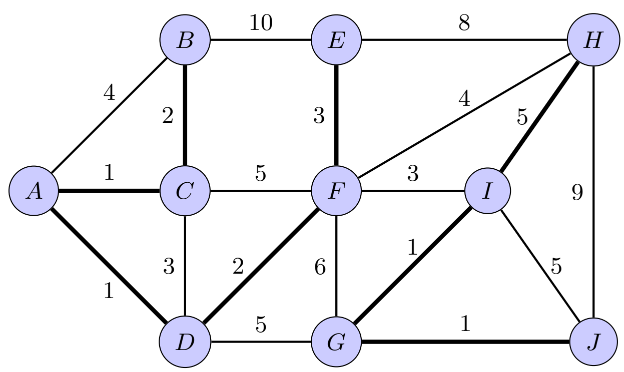
points in T. The symmetric difference of the edge sets of these paths is a T-join of cost at most $w(M)$ (smaller if the paths are not edge-disjoint because the costs are positive). Conversely, every T-join F in G contains edge disjoint paths pairing up the points in T and hence gives rise to a perfect matching M in H whose weight is at most the cost of F (smaller if F contains edges outside the paths, as the costs are positive). Hence, any minimum weight perfect matching in H gives rise to edge disjoint paths in G and the union of their edges form a minimum cost T-join.
The weight function w for the given graph can be computed with Dijkstra's algorithm and are given in the following table:

From	**To**	**Cost**	**Path**
B	E	9	(B, C, A, D, F, E)
B	H	10	(B, C, A, D, F, H)
B	J	10	(B, C, A, D, G, J)
E	H	7	(E, F, H)
E	J	8	(E, F, I, G, J)
H	J	7	(H, I, G, J)

■

[6pts] (c) Solve the matching problem and give a minimum cost walk from A to H traversing every edge of the graph at least once.

Solution. The unique minimum weight perfect matching is $\{BE, HJ\}$ (trying all 3 possibilities). This corresponds to taking paths (B, C, A, D, F, E) and (H, I, G, J). Doubling the edges on these two paths, we obtain the following graph (double edges are thick).



An Eulerian walk from A to H can be easily found by first deleting the double edges and finding an Eulerian walk from A to H in the resulting graph. This can be done since the resulting graph is connected. In fact there is only one such walk: $(A, B, E, H, F, C, D, G, F, I, J, H)$. Then adding in the double edges by traversing then in both directions gives

$$(A, C, A, D, A, B, C, B, E, F, E, H, I, H, F, D, F, C, D, G, J, G, I, G, F, I, J, H)$$

The sum of the cost of the edges of the original grapg equals 78. The cost of the two T-join found is $7 + 9 = 16$, hence the total walk has length $78 + 16 = 94$. ■