Exam Advanced Probability TW 3560
4th June 2017, 13:30-16:30

- The exam is a closed book exam. You may use a simple non-graphical calculator.
- All solutions should be well-documented and explained.
- In the **first part** there are 5 questions. Every correct answer gives 1 point, you can reach maximally 5 points.
- The **second part** will be considered if in the first part the student scored more than 3 points. For the second part the points are distributed as follows:

	Exercise 1	Exercise 2	Exercise 3	Exercise 4
Points	2-1-1-2	3-2-2-2	2-3-2	1-2-2-3

- The total number of points is 35. The grade is calculated in the following way:

$$\text{grade} = \min\left(\frac{1}{3.5}(\sharp\text{points}) + \text{bonus point}, 10\right).$$

Part I (5 P.)
Indicate if the following statements are true or false and explain why.

1. Let X be a non-negative random variable then the expected value is defined as
$$\mathbb{E}(X) = \inf\{\mathbb{E}(Y); Y \leq X; Y \text{ simple}\}.$$

2. Let $(X_n)_{n\geq 0}$ be a Markov chain on some countable space E and transition matrix $P = (p(x,y))_{x,y\in E}$ then
$$p^n(x,\cdot) = \mathbb{P}_x \circ X_n^{-1}.$$

3. Let $X \underset{n\to\infty}{\longrightarrow} X$ in probability and $X \underset{n\to\infty}{\longrightarrow} 1$ in distribution then $X \neq 1$ a.s.

4. Consider a Markov chain $(X_n)_{n\geq 0}$ on E with some initial distribution ν and transition probabilities $p(x,y)$ for all $x, y \in E$. Then for all $x, y \in E$
$$p^{102}(x,y) = \sum_{k\in E} p^3(x,k)p^{99}(k,y).$$

5. The characteristic function φ_X of a normal variable with mean 2 and variance 1 is
$$\varphi_X(t) = e^{-\frac{t^2}{2}+2it}.$$

Part II

Exercise 1 (6 P.)
Let $(X_n)_{n\geq 0}$ be a Markov chain with transition matrix

$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

1. Find all stationary distributions and draw the transition diagram.
2. The chain starts in $i = 1$. What is the expected number of steps before it returns to 1?
3. Compute $\mathbb{P}(X_4 = 1, X_2 = 1 | X_0 = 3)$.
4. Each time the chain visits the state $i = 1$, 1 Euro is added to an account and 2 Euros for visiting the state $i = 2$ and nothing in the state 3. Estimate the amount of money on the account after 1000 transitions.
   (Hint: You may use the law of large numbers for Markov chains as a good approximation: Let S be a finite state space, $(X_n)_{n\geq 0}$ an irreducible Markov chain and f a bounded function. Then
   $$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(X_n) = \sum_{i\in S} f(i)\alpha(i)$$
   where α is the stationary distribution.)

Exercise 2 (9 P.)
Let $(Z_n)_{n\geq 1}$ be a sequence of independent random variables such that for $j = 1, 2, \ldots$ we have for some constant $a \in (1, 1.5)$

$$\mathbb{P}(Z_j = j^a) = \mathbb{P}(Z_j = -j^a) = \frac{1}{6} j^{-2(a-1)}$$

and

$$\mathbb{P}(Z_j = 0) = 1 - \frac{1}{3} j^{-2(a-1)}.$$

1. State and verify the Lyapunov conditions.
2. Does there exist sequences of real numbers $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ such that
   $$\frac{\sum_{j=1}^{n} Z_j - a_n}{b_n} \underset{n\to\infty}{\longrightarrow} Z$$
   in distribution where $Z \sim N(0, 1)$?
3. Let now $(Z_n)_{n\geq 1}$ be i.i.d. random variables. State the assumptions of the classical central limit theorem (CLT) for i.i.d. random variables.
4. Show that the classical CLT implies the CLT of Lindeberg-Lévy-Feller.

Exercise 3 (7 P.)
A car insurance provides collision coverage above 300 Euros deductible (amount which has to be paid by the insurerer) up to a maximum of 2500 Euros. Let X denote the amount of possible collision damage of the car. Assume that X has density

$$f(x) = \frac{2}{(2+x)^2} \mathbb{1}_{x \geq 0}.$$

1. Write the payout Y of an insurance policy per policy claim in terms of X.
2. Determine the density of Y and compute the expected payout.
3. We consider a second policy which provides a standard payout of 200 Euros if there are more than 2 claims in a period and Y if there is 1 claim in a period. The probability that there is 1 claim in a period is equal to $\frac{2}{3}$ and more than 1, $\frac{1}{3}$.
   Compute the expected payout in a period. Which policy provides the larger average payout?

Exercise 4 (8 P.)
Let $(X_n)_{n\geq 1}$ be a sequence of independent exponential random variables with parameter $\alpha > 0$ and $(Y_n)_{n\geq 1}$ a sequence of independent uniform random variables on $[0, 1]$.

1. Let $\epsilon > 0$, compute $\mathbb{P}(\alpha X_n > \epsilon \log(n))$.
2. Show that

$$\mathbb{P}(\alpha X_n > \epsilon \log(n)) = \begin{cases} 0 & \text{if } \epsilon > 1 \\ 1 & \text{if } \epsilon \leq 1. \end{cases}$$

3. Prove that almost surely

$$\limsup_{n\to\infty} \frac{X_n}{\log(n)} = \frac{1}{\alpha}.$$

4. Denote by $M_n := \min(Y_1, ..., Y_n)$. Show that in distribution

$$nM_n \underset{n\to\infty}{\longrightarrow} W$$

   where $W \sim Exp(1)$.