Exam Advanced Probability TW 3560
19th April 2017, 13:30-16:30

- The exam is a closed book exam. You may use a simple non-graphical calculator.
- All solutions should be well-documented and explained.
- In the end you will find a table with common characteristic functions which you may use.
- Exercises with a * are small brain teasers and not necessary for solving the whole exercise, skip them if you get stuck.
- In the **first part** there are 5 questions. Every correct answer gives 1 point, you can reach maximally 5 points.
- The **second part** will be considered if in the first part the student scored more than 3 points. For the second part the points are distributed as follows:

	Exercise 1	Exercise 2	Exercise 3	Exercise 4	Exercise 5
Points	1-2-4	2-2-2-1-1-1	3-2-2-1-2	1-2-2-1-2-1	4

- The total number of points is 40+4. The grade is calculated in the following way: grade=min($\frac{1}{4}$ * ♯ points+ bonus, 10).

Part I (5 P.)
Indicate if the following statements are true or false and explain why.

True

1. Let $f(x) = 5^{-x}$ and $X \sim Geo(0.5)$ is a geometric random variable with parameter $p = \frac{1}{2}$, then $= \left(\frac{1}{5}\right)^x$
$$\mathbb{E}(f(X)) = \sum_{k=1}^{\infty} \frac{1}{10^k}.$$

True

2. Let $p \geq 1$ and $X_n \to X$ in L_p as $n \to \infty$ then for all $t \in \mathbb{R}$, $\varphi_{X_n}(t) \to \varphi_X(t)$ as $n \to \infty$ where φ_X is the characteristic function of the random variable X.

False

3. Let $X_1, X_2...$ be a sequence of i.i.d. random variables such that for all $i \in \mathbb{N}$, $\mathbb{E}(X_i) = 2$ and $\mathbb{E}(X_i^2) = 9$, then
$$\frac{X_1 + ... + X_n - 2n}{\sqrt{n}} \xrightarrow{d} X$$
as $n \to \infty$ where $X \sim N(0, 5)$.

False

4. If α is a stationary distribution for a Markov chain with transition matrix P, then we do not necessarily have that $\alpha P = \alpha$.

False

5. Let $(A_n)_{n\geq 1}$ be a sequence of events. Then
$$\limsup_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m.$$

Part II

Exercise 1 (7 P.)

A factory is developing wheel bearings (wiellagers) for cars. The failure time T_A of bearing A has density

$$f(x) = 4e^{-4x}\mathbb{1}_{x\geq 0}$$

and the failure time T_B of bearing B has distribution

$$\mathbb{P}(T_B = k) = \frac{1}{3}\left(\frac{2}{3}\right)^{k-1}$$

for $k \geq 1$. Before going into production the bearings have to pass a stress test. During the stress test of the bearing an (invasive) measurement is taken at time points 1 and 2 and there is some chance that the bearing breaks during the measurement.

Let Z_A and Z_B denote the failure times of bearing A and B after the stress test. Experience has shown that

$$Z_A = \begin{cases} 1 & \text{with probability } 0.1 \\ 2 & \text{with probability } 0.2 \\ T_A & \text{with probability } 0.7 \end{cases}$$

and

$$Z_B = \begin{cases} 1 & \text{with probability } 0.1 \\ 2 & \text{with probability } 0.1 \\ T_B & \text{with probability } 0.8. \end{cases}$$

1. Determine the distributions μ_A and μ_B of Z_A respectively Z_B. What type of distribution is it?
2. Compute the expected values of Z_A and Z_B.
3. Application: compute the maintenance cost functions K_A resp. K_B of bearing A and B where
$$K_i = \mathbb{E}(2^{-Z_i})$$
and $i \in \{A, B\}$. Imagine you are a manager and have to decide which bearing to produce, which one would you choose? Reken later uit.

Exercise 2 (9 P.)

Let $(X_n)_{n\geq 1}$ be a sequence of independent Gaussian random variables with distributions $X_n \sim N(m_n, \sigma_n^2)$ where for all $n \geq 1$, $|m_n| < \infty$, $\sigma_n^2 \in (0, \infty)$.

1. State and verify the Lyapunov condition for $(X_n)_{n\geq 1}$.
(Hint: You can use Cauchy-Schwartz on the forth moment.)
2. Find $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ such that
$$Y_n := \frac{\sum_{k=1}^{n} X_k - a_n}{b_n} \xrightarrow{d} Z \tag{1}$$
where $Z \sim N(0,1)$ as $n \to \infty$.
3. Assume that $|m| < \infty$, $\sigma^2 \in (0, \infty)$ and $X \sim N(m, \sigma^2)$. Show that
$$X_n \xrightarrow{d} X \Leftrightarrow \lim_{n\to\infty} m_n = m, \ \lim_{n\to\infty} \sigma_n^2 = \sigma^2. \tag{2}$$

4. Is the previous equivalence (2) true for general distributions of X_n?

5.* Does the sequence $(Y_n)_{n\geq 1}$ in (1) converge almost surely, in L_p (for some $p \geq 1$) or in probability towards Z, where $Z \sim N(0,1)$ for any $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$? Prove or disprove the claim.
(Hint: For disproving the claim it is enough to show that one specific convergence does not hold. Which one?)

6.* Are the sequences $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ in (1) unique? Prove or disprove the claim. no normal convergence!

Exercise 3 (10 P.)
In working with a particular gene for fruit flies, geneticists classify an individual fruit fly as *dominant GG*, *hybrid Gg* or *recessive gg*. From one generation to the other the gene is splitting and recombining. The gene expression of an offspring in each generation only depends on the parents generation. → Markov property.

Assume that a geneticist is experimenting with different crossings of fruit flies over generations n and is recording the resulting gene expression X_n.

The rules of inheritance as determined by the monk Gregor Mendel (excluding the possibility of mutations) are as follows. The offspring of two dominant parents GG will be dominant in 50% of the cases and hybrid Gg in 50% of the cases, whereas the offspring of a recessive pair gg is with equal probability either hybrid or recessive. Finally the offspring of a hybrid is a hybrid in 50% of the cases and dominant in 25% of the cases.

1. Define an appropriate state space E, transition matrix P and draw the transition diagram for the Markov chain $(X_n)_{n\geq 0}$.

2. What is the probability the third generation offspring is dominant given the first generation offspring is recessive?

3. If the population of fruit flies initially is 20% dominant, 50% hybrid and 30% recessive, what percentage of the population is dominant after 3 generations? You can use that

$$P^3 = \begin{bmatrix} 5/16 & 1/2 & 3/16 \\ 1/4 & 1/2 & 1/4 \\ 3/16 & 1/2 & 5/16 \end{bmatrix}$$

(where 1=dominant 2=hybrid and 3=recessive).

4. Is the Markov chain irreducible?

5. Determine the stationary distribution(s) of the Markov chain. Is it unique? What is the mean first return time to the dominant state?

Exercise 4 (9 P.)
Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d random variables with common density f. Define for $s > 0$

$$\tau(s) = \inf\{n \geq 1; X_n > s\}$$

the index of the first X_n-random variable which is exceeding a certain level s.

1. Let $k \in \mathbb{N}$, express the event $\{\tau(s) = k\}$ in terms of $X_1, ..., X_k$.

2. Compute $\mathbb{P}(\tau(s) = k)$ for $k \in \mathbb{N}$ in terms of $p_s := \mathbb{P}(X_1 > s)$. Identify the distribution of $\tau(s)$.

3. Show that the characteristic function of $p_s\tau(s)$ is equal to

$$\varphi_{p_s\tau(s)}(t) = \frac{p_s e^{itp_s}}{1-(1-p_s)e^{itp_s}}.$$

4.* Prove that

$$\lim_{s\to\infty} \frac{p_s e^{itp_s}}{1-(1-p_s)e^{itp_s}} = \frac{1}{1-it}$$

and identify the limiting distribution. Gamma (1, 1).

5. Application: every day a drug is given to a patient and is decomposed in the body within one day. X_n measures the concentration of the drug in the body on day n. If the concentration exceeds 10 then the drug is dangerous for the patient and he/she needs to be taken to the ICU. Assume now that the common density of $(X_n)_{n\geq 1}$ is given by

$$f(x) = \frac{1}{(1+x)^2}\mathbb{1}_{x\geq 0}.$$

What is the probability that the patient will be moved to the ICU for the first time after 7 days of treatment?

6. Find the maximal $k \geq 1$ such that $\mathbb{P}(\tau(10) > k) \geq 0.8$. What does the result mean in combination with (5)?

Extra exercise 5* (4 P.)
Let $X_1, X_2, \ldots$ be i.i.d. with $\mathbb{E}(|X_1|^{2+\delta}) < \infty$ for some $\delta > 0$. Let $M_n = \max(X_1, \ldots, X_n)$, prove that

$$\frac{1}{n}M_n \xrightarrow{a.s.} 0$$

as $n \to \infty$.

Some specific characteristic functions:

Distribution	Notation	Characteristic function
Uniform	$U[a,b]$	$\frac{e^{itb}-e^{ita}}{it(b-a)}$
Exponential	$Exp(\lambda)$	$\frac{\lambda}{\lambda-it}$
Gamma	$\Gamma(p,\theta)$	$\left(\frac{1}{1-\theta it}\right)^p$
Normal	$N(m,\sigma^2)$	$e^{itm-\frac{1}{2}t^2\sigma^2}$
Cauchy	$C(0,1)$	$e^{-\lvert t\rvert}$
One point	δ_a	e^{ita}
Bernoulli	$B(p)$	$(1-p)+pe^{it}$
Binomial	$Bin(n,p)$	$((1-p)+pe^{it})^n$
Geometric	$Geo(p)$	$\frac{pe^{it}}{1-(1-p)e^{it}}$
Poisson	$Poi(\lambda)$	$e^{\lambda(e^{it}-1)}$