Exam Advanced Probability TW 3560
14th April 2016, 9:00-12:00

- The exam is a closed book exam. You may use a simple non-graphical calculator.
- All solutions should be well-documented and explained.
- In the **first part** there are 5 questions. Every correct answer gives 1 point, you can reach maximally 5 points.
- The **second part** will be considered if in the first part the student scored more than 3 points. For the second part the points are distributed as follows:

	Exercise 1	Exercise 2	Exercise 3	Exercise 4
Points	2-2-2-3-2	4-2-1-1	2-2-2-3-2-2-1	3-1-2-3-2-1

- The total number of points is 50. The grade is calculated in the following way: grade=max($\frac{1}{5} * \sharp$ points+ bonus, 10).

Part I

Indicate if the following statements are true or false and explain why.

1. Let $A_1, ..., A_n, .. \in \mathscr{F}$ denote a sequence of events defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$. If $\sum_{n=0}^{\infty} \mathbb{P}(A_n) = \infty$ then A_n appears infinitely many times almost surely.
2. If X_n converges to X in distribution, then for every f Borel-measurable function $\mathbb{E}(f(X_n)) \to \mathbb{E}(f(X))$ as $n \to \infty$.
3. If there exists a unique stationary distribution for a Markov chain, then the chain is irreducible.
4. Chebyshevs inequality is a consequence of Markov's inequality.
5. Convergence almost surely implies convergence in distribution.

Part II

Exercise 1

A country has $m+1$ cities, $m \in \mathbb{N}$, of which one is the capital. There is a direct railway connection between each city and the capital but there are no tracks between 2 "non-capital" cities. A traveler starts in the capital and takes a train to a randomly chosen non-capital city (all "non-capital" cities are chosen with equal probability), spends a night there, returns the next morning and immediately boards the train to the next city according to the same rule, spends the night there, etc. We assume that his choice of the city is independent of the cities visited in the past. Let $(X_n)_{n\in\mathbb{N}}$ denote the number of different visited non-capital cities up to and including day n, so that $X_0 = 1$ and $X_1 = 1$ or $X_1 = 2$ etc.

1. Explain why $(X_n)_{n\in\mathbb{N}}$ is a Markov chain, find the appropriate state space S and the transition probabilities.
2. Classify the states into recurrent, transient, periodic. Sketch a transition graph for $m = 3$.
3. Calculate for $m = 3$ the distribution $\mathbb{P}(X_2 = i)$ for $i \in \{1, 2, 3\}$ if $\mathbb{P}(X_0 = 1) = \mathbb{P}(X_0 = 2) = \frac{1}{2}$.
4. Let τ_m be the first time the traveler visited all non-capital cities, i.e.
$$\tau_m = \inf\{n \in \mathbb{N} : X_n = m\}$$
What is the distribution of τ_m for $m = 1$, $m = 2$?
5. Compute $\mathbb{E}(\tau_m)$ for general m. (Hint: You may find a geometric random variable which is modeling the waiting time until the first success.)
6. What are stationary solution(s) π? Do we have here that $\lim_{n\to\infty} \mathbb{P}(X_n = i) = \pi_i$ for all $i \in S$?

Exercise 2:

1. Let X be a standard normal distribution. Show the upper and lower bound for $x \geq 0$
$$\frac{1}{\sqrt{2\pi}} \left(\frac{1}{x + \frac{1}{x}} \right) e^{-\frac{x^2}{2}} \leq \mathbb{P}(X \geq x) \leq \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-\frac{x^2}{2}}.$$

2. Consider a sequence of i.i.d. random variables $X_1, ..., X_n, ..$ such that $X_i \sim N(0,1)$. Let $S_n = \sum_{i=1}^n X_i$. Show that there exist a sequence of real numbers $(\epsilon_n)_n$ such that $\epsilon_n \to 0$ as $n \to \infty$ such that $\forall x > 0$:

$$\mathbb{P}(S_n \geq xn) = (1+\epsilon_n)\frac{1}{\sqrt{2\pi n}}e^{-n\frac{x^2}{2}}.$$

3. Deduce that for every $x > 0$:

$$\lim_{n\to\infty}\frac{1}{n}\log(\mathbb{P}(S_n \geq nx)) = -\frac{x^2}{2}.$$

4. Let us consider now the cumulant generating function $\Lambda(t) = \log(\mathbb{E}(e^{tX_1}))$. Show that it's Legendre transform defined by

$$\Lambda^*(x) := \sup_{t\in\mathbb{R}}(t * x - \Lambda(t))$$

is equal to $\frac{x^2}{2}$ for all $x > \mathbb{E}(X_1)$.

Exercise 3:

Let us consider $X_1, X_2, ..$ i.i.d. random variables such that $\mathbb{P}(X_i = -1) = \mathbb{P}(X_i = 1) = \frac{1}{2}$. We want to show that for $S_n = \sum_{i=1}^n X_i$:

$$\lim_{n\to\infty}\frac{1}{n}\log(\mathbb{P}(S_n \geq nx)) = -I(x). \tag{1}$$

where

$$I(z) = \begin{cases} \frac{1+z}{2}\log(1+z) + \frac{1-z}{2}\log(1-z) & \text{if } z \in [-1,1] \\ \infty & \text{if } |z| > 1. \end{cases} \tag{2}$$

(Remark: We agree that $0\log(0) := 0$.)

1. Argue that the claim (1) with I given in (2) is trivially true for the special cases $x = 0$, $x = 1$ and $x > 1$.

2. Consider now $x \in (0,1)$. Determine the distribution of $\frac{S_n+n}{2}$ and show that

$$\mathbb{P}(S_n \geq xn) = 2^{-n}\sum_{k\geq(1+x)n/2}\binom{n}{k}.$$

3. Call now $a_n(x) := \lceil n(1+x)/2 \rceil$ and $Q_n(x) := \max\left\{\binom{n}{k} : a_n(x) \leq k \leq n\right\}$. Demonstrate that

$$2^{-n}Q_n(x) \leq \mathbb{P}(S_n \geq nx) \leq (n+1)2^{-n}Q_n(x).$$

4. Use Stirling's formula $\lim_{n\to\infty} \frac{1}{n!}n^n e^{-n}\sqrt{2\pi n} = 1$ to obtain

$$\lim_{n\to\infty} \frac{1}{n} \log(Q_n(x)) = -I(x) + \log(2)$$

   with I defined in (2).

5. Conclude the proof of the claim (1). What does it mean asymptotically for large n?

6. Show that the Legendre transform of the cumulant generating function is equal to $I(x)$ defined in (2) for all $x > \mathbb{E}(X_1)$.
   (Hint: You may use that $\tanh^{-1}(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ for $z \in (-1,1)$ and $\cosh(\tanh^{-1}(z)) = \frac{1}{\sqrt{1-z^2}}$.)

7. Application: Consider an insurance company which settles a fixed number of claims (say 1 per day) and receives a steady income from premium payments each day. The sizes of the claims are random. The company wants to minimize the risk that at the end of the month the amount of claims realized is larger than the income. The large claims can create problems, they appear with probability $\frac{1}{2}$ independent of each other. What is the probability that in April there are more than 15 large claims?

Exercise 4:

We want to estimate the energy consumption of a two factories. The first one is using a large number of devices of type A. These devices can run in 3 different states and the energy consumption in each state is different. If the device is running in state 1 which happens in average half of the time then the consumption can be approximated by a normal random variable with mean 0 and variance σ^2. In state 2 it is exponentially distributed with mean 4 which happens $\frac{1}{3}$ of the time and finally in the third state the consumption if geometrically distributed with parameter $\frac{1}{3}$. (Recall that a random variable on $\{1, 2, ..\}$ is geometrically distributed with parameter p if $\mathbb{P}(X = k) = p(1-p)^{k-1}$.)

1. What is the law of the total energy consumption of one device X depending on σ^2? Show that the mean is $\frac{11}{6}$ and the variance is $\frac{1}{2}\sigma^2 + \frac{353}{36}$.

2. Consider now $S_n = \sum_{i=1}^{n} X_i$ where X_i denotes the energy consumption of device i. Assume that they run independent of each other. Does S_n has a limit in distribution for large n w.r.t. some appropriate rescaling?

3. Approximate the probability that S_{100} exceeds 200 for $\sigma = 5$.

4. Estimate the error of this approximation.

A second factory uses other devices for their production. Their energy consumption for n devices can be modeled by the following random variable

$$Y_n = Z + \delta_1 + \delta_{1/4} + ... + \delta_{1/n^2}$$

where $Z \sim N(0, \sigma^2)$. Determine the law of Y_n.

5. Show that almost surely Y_n converges to W as $n \to \infty$, where $W \sim N(\frac{\pi^2}{6}, \sigma^2)$. (Hint: You can use that $\sum_{i=1}^{\infty} i^{-2} = \frac{\pi^2}{6}$.)

6. Assume that factory 1 runs 20 devices of type A and factory 2 runs 30 devices of type B. Imagine that you work for the city and have to decide to give a financial bonus to the company which is more environmental friendly meaning that it consumes less energy than the other. To which factory would you give the bonus?