Resit Exam Advanced Probability TW 3560
8th July 2015, 9:00-12:00

- The exam is a closed book exam. You may use a simple non-graphical calculator.
- Write clear and motivate your answers.
- The total number of points is 35. The grade is calculated in the following way: grade=$\frac{10}{35}*$ number of reached points.

Part I
Indicate if the following statements are true or false and explain why.

1. (1p) If $\mathscr{F}$ is an sigma-algebra then it is also a semi-algebra.
2. (1p) If $\Omega = \mathbb{Q}$ then $\mathscr{P}(\Omega) = \mathscr{F}$ with appropriate $\mathbb{P}$ defines a probability triple.
3. (1p) Not every random variable can be approximated via simple functions.
4. (1p) If a sequence of random variables $(X_i)_i$ is i.i.d. then the empirical average $\frac{1}{n}(X_1 + ... + X_n)$ converges.
5. (1p) The moment generating function of a random variable always exist.

Part II

Exercise 1: A resistant virus can exist in 3 different strains A, B or C. If the virus is in strain A or B it can mutate to strain A, B or C with equal probability. If the virus is in strain C it cannot mutate. There exists treatment for treating the virus strain A or B but not for C.

1. (1p) Draw the diagram.
2. (2p) Is the chain irreducible? Argue why or why not.
3. (2p) Calculate the transition matrix after n steps P^n.
4. (3p) Prove weather the states are transient or recurrent and determine their period.
5. (1p) What is (are) stationary solution(s) π?

6. (1p) Does $\lim_{n\to\infty} \mathbb{P}(X_n = A) = \pi(A)$? What is the probability that in the long run a person with this virus can be treated?

7. (2p) Show that the mean return time to state A, m_A is equal to 3. Does it contradict the previous statement?

Exercise 2: TU Delft decided to have a summer cleaning and is moving some old things out of the building. They can use use just one elevator which can carry 900 kg. The boxes are not packed in a uniform and independent way. The moments of the weights X_j of boxes $j \in \mathbb{N}$ (rescaled by a factor of 1000) are given by

- $\mathbb{E}(X_j) = e^{-j}$ for all $j \in \mathbb{N}$
- $\mathbb{V}(X_j) = 1 + \frac{1}{j^2}$ for all $j \in \mathbb{N}$
- $Cov(X_j, X_k) \leq e^{-|j-k|}$ for all $j, k \in \mathbb{N}$ with $j \neq k$.

Show for $Y_n = \frac{1}{n}\sum_{j=1}^{n} X_j$

1. (1p) $\lim_{n\to\infty} \mathbb{E}(Y_n) = 0$ and $\lim_{n\to\infty} \mathbb{V}(Y_n) = 0$. (Hint: You can use that $\mathbb{V}(X_1 + ... + X_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} Cov(X_i.X_j)$).

2. (2p) Show Y_n converges in probability towards 0.

3. (2p) Assume now that $(X_j)_j$ are normally distributed and independent. Show that $\frac{1}{\sqrt{n}}\sum_{j=1}^{n} X_j$ converges in distribution towards a standard normal variable Z.

4. (1p) Assume we can approximate the total average weight by $Z\mathbb{1}_{Z\geq 0}$. What is the probability that the movers can transport all packages at once? (Remeber the rescaling.)

5. (1p) Can you apply the Berry-Esséen bound? Why or why not?

Exercise 3: Adam has N batteries available for a electrical device. Once the battery is empty he replaces it with another one. The lifetime L_i of one battery i is independent of the lifetime L_j of battery j $(i \neq j)$ and is exponential distributed with mean 2 (rescaled by 100 hours).

1. (1p) What is the probability density function of $L_1 + L_2$?

2. (1p) Let N be a random variable with sample space $\{1, 2, 3\}$ and distribution

$$\mathbb{P}(N = 2) = \mathbb{P}(N = 3) = \frac{1}{4},\ \mathbb{P}(N = 1) = \frac{1}{2}$$

What is $\mathscr{L}(X_N)$, the law of $X_N = \sum_{i=1}^{N} L_i$?

3. (2p) What is the expected value of X_N?

4. (1p) Give an non-trivial upper bound on the probability that the device works at least 1000 hours?

5. (2p) Berta has an analogous device and 2 batteries at home. What is the probability that Bertas device will work longer Adams device with N distributed as in (2)?

Exercise 4:

1. (2p) Let Y be uniformly distributed on $[0, 1]$. Define

$$Z_n^{(1)} = \mathbb{1}_{Y \leq 1/n}.$$

Does $Z_n^{(1)}$ converge towards 0 in probability? Does it converge almost surely?

2. (2p) Let $(Y_n)_n$ be a sequence of independent random variables, all uniform on $[0, 1]$. Define further

$$Z_n^{(2)} = \mathbb{1}_{\{Y_n \leq 1/n\}}.$$

Does $Z_n^{(2)}$ converge towards 0 in probability? Does it converge almost surely?