Solutions Resit- Advanced Probability

TW 3560

July 10, 2015

Part I

1. Yes, because if $\mathscr{F}$ is a sigma-algebra then it is also a semi algebra (closed under finite intersection).
2. Yes, if the sample space is countable we can choose the power set as sigma algebra.
3. No, we defined a general random variable via limits of simple functions.
4. No, we need that the mean is finite and the variance is bounded or the forth moment is finite.
5. No, it can be infinite even if the mean of the random variable is finite. The characteristic function always exist.

Part II

Exercise 1.1:

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix}$$

1.2:

The chain is not irreducible because from strain C it is not possible to go to any of the other strains.

1.3:

$$P^2 = \begin{pmatrix} \frac{2}{9} & \frac{2}{9} & \frac{5}{9} \\ \frac{2}{9} & \frac{2}{9} & \frac{5}{9} \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$P^3 = \begin{pmatrix} \frac{4}{27} & \frac{4}{27} & \frac{19}{27} \\ \frac{4}{27} & \frac{4}{27} & \frac{19}{27} \\ 0 & 0 & 1 \end{pmatrix}$$

we have to show via induction (!!) that

$$P^n = \begin{pmatrix} \frac{2^{n-1}}{3^n} & \frac{2^{n-1}}{3^n} & 1 - \frac{2^{n-1}}{3^n} \\ \frac{2^{n-1}}{3^n} & \frac{2^{n-1}}{3^n} & 1 - \frac{2^{n-1}}{3^n} \\ 0 & 0 & 1 \end{pmatrix}$$

1.4: All states are aperiodic. We will show that states A, B are transient and C is recurrent. $\sum_{n=1}^{\infty} P_{AA}^n < \infty$ and A and B are in one class hence both are transient. $\sum_{n=1}^{\infty} P_{CC}^n = \infty$ hence C is recurrent.
1.5: We solve the system and get that the unique distribution is equal to $(\pi_1, \pi_2, \pi_3) = (0, 0, 1)$.
1.6: $\lim_{n\to\infty} \mathbb{P}(X_n = A) = \frac{2^{n-1}}{3^n} = 0$. The probability of being able to be treated in the long run is 0.
1.7: We sum $m_A = \sum_{k=1}^{\infty} kP_{AA}^k = 3$. Theorem 8.4.9. states that $\pi_A = 1/m_A$ for irreducible Markov chains. Since the chain is not irreducible there is no contradiction.
2.1:

$$0 \leq \lim_{n\to\infty} \mathbb{E}(Y_n) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}(X_j) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} e^{-j} \leq \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{\infty} e^{-j} = 0$$

and since the series $\sum_{j=1}^{\infty} e^{-j} = 1/1 - e^{-}1$ is convergent we can take the limit and it is 0.

$$0 \leq \mathbb{V}(Y_n) = \frac{1}{n^2}\mathbb{V}(X_1 + ... + X_n) = \frac{1}{n^2} \sum_{j=1}^{n} \mathbb{V}(X_j) + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} Cov(X_i, X_j)$$
$$\leq \frac{1}{n^2} \sum_{j=1}^{n} (1 + \frac{1}{j^2}) + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} e^{-|i-j|}$$

For the first term we observe

$$\frac{1}{n^2} \sum_{j=1}^{n} (1 + \frac{1}{j^2}) \leq \frac{1}{n} + \frac{1}{n^2} \sum_{j=1}^{\infty} \frac{1}{j^2} \leq \frac{1}{n} + \frac{C}{n^2} \overset{n\to\infty}{\longrightarrow} 0$$

since $\sum_{j=1}^{\infty} 1/j^2 < \infty$, for the second

$$\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} e^{-|i-j|} \leq \sum_{i=1}^{n} \sum_{j=1}^{\infty} e^{-j} = \frac{C}{n} \overset{n\to\infty}{\longrightarrow} 0$$

2.2: Let $\epsilon > 0$, since $Y_n \geq 0$

$$\mathbb{P}(Y_n > \epsilon) \leq \frac{\mathbb{E}(Y_n)}{\epsilon} \leq \frac{\sum_{j=1}^{\infty} e^{-j}}{\epsilon n} \overset{n\to\infty}{\longrightarrow} 0$$

via Markov inequality.
2.3: Notice that we cannot use the CLT because the means are variances are NOT equal. We calculate the characteristic function and show that it converges towards $e^{-t^2/2}$. By theorem 11.1.14 the claim follows. From independence we have

$$\varphi_n(t) = \mathbb{E}(e^{it\frac{1}{\sqrt{n}}\sum_{j=1}^{n} X_j}) = \prod_{j=1}^{n} \mathbb{E}(e^{it\frac{X_j}{\sqrt{n}}})$$

We use that if $X_j \sim N(e^{-j}, 1 + \frac{1}{j^2})$ then $\frac{X_j - e^{-j}}{\sqrt{1+j^{-2}}} \sim N(0, 1)$. Hence

$$\mathbb{E}(e^{it\frac{X_j}{\sqrt{n}}}) = e^{it\frac{e^{-j}}{\sqrt{n}}} \mathbb{E}(e^{it\frac{\sqrt{1+j^{-2}}}{\sqrt{n}} Z}) = e^{it\frac{e^{-j}}{\sqrt{n}}} e^{-\frac{1}{2}\frac{t^2(1+j^{-2})}{n}}$$

so

$$\prod_{j=1}^{n} \mathbb{E}(e^{it\frac{X_j}{\sqrt{n}}}) = e^{\frac{it}{\sqrt{n}}\sum_{j=1}^{n} e^{-j}} e^{-\frac{t^2}{2}\frac{1}{n}\sum_{j=1}^{n}(1+j^{-2})}$$

since the series $\sum_{j=1}^{\infty}(1+j^{-2})$ and $\sum_{j=1}^{\infty} e^{-j}$ are converging we deduce that the right hand side converges towards $e^{-t^2/2}$.

2.4: Since we have that $N(0,1)$ we can look in the tables and compute (it is enough to remark that) We look for

$$\Phi(0.9) - \Phi(0) = 0.79 - 0.5 = 0.29.$$

2.5: No, because the X_j have to be i.i.d.

3.1: We have that $L_i \sim Exp(1/2)$ with pdf f_1. Let f_2 be the pdf of $L_1 + L_2$.

$$f_2(z) = \int_{-\infty}^{\infty} f_1(x)f_1(z-x)dx = \int_0^z \frac{1}{4}e^{-1/2z}dx = \frac{1}{4}ze^{-1/2z}$$

3.2: Let B be a measurable set.

$$\mathbb{P}(X_N \in B) = \frac{1}{2}\mathbb{P}(L_1 \in B|N=1) + \frac{1}{4}\mathbb{P}(L_1+L_2 \in B|N=2) + \frac{1}{4}\mathbb{P}(L_1+L_2+L_3 \in B|N=3)$$

hence $\mathscr{L}(X_N) = \frac{1}{2}\mathscr{L}(L_1) + \frac{1}{4}\mathscr{L}(L_1+L_2) + \frac{1}{4}\mathscr{L}(L_1+L_2+L_3)$

3.3: Let f_3 be the pdf of $L_1 + L_2 + L_3$

$$f_3(z) = \int_0^z f_1(x)f_2(z-x)dx = \frac{1}{16}z^2e^{-1/2z}$$

$$\mathbb{E}(X_N) = 1 + \frac{1}{4}\int_0^{\infty}\frac{1}{4}z^2e^{-1/2z}dz + \frac{1}{4}\int_0^{\infty}\frac{1}{16}z^3e^{-1/2z}dz = \frac{7}{2}$$

3.4:

$$\mathbb{P}(X_N > 10) \le \frac{7}{20} = 0.35$$

3.5: Let B be the lifetime of Bertas device, B and X_N are independent.

$$\begin{aligned}\mathbb{P}(B > X_N) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbb{1}_{x>y} f_2(x)(\frac{1}{2}f_1(y) + \frac{1}{4}f_2(y) + \frac{1}{4}f_3(y))dxdy \\ &= \int_0^{\infty}(\frac{1}{2}f_1(y) + \frac{1}{4}f_2(y) + \frac{1}{4}f_3(y))\int_y^{\infty} f_2(x)dxdy = \frac{37}{64} = 0.57\end{aligned}$$

with

$$\int_y^{\infty} f_2(x)dx = \frac{1}{2}e^{-1/2y}(2+y).$$

4.1: For all $\epsilon > 0$

$$\mathbb{P}(Z_n^{(1)} > \epsilon) \le \mathbb{P}(Z_n^{(1)} = 1) = \mathbb{P}(Y \le \frac{1}{n}) = \frac{1}{n} \to 0$$

as n is going to infinity, hence it converges to 0 in probability. It also converges a.s. towards 0. The sequence is of the form $1, 1, 1, 0, 0, 0,$ where the transition point $n = 1/Y(w)$ is is finite. $Y(w) = 0$ happens with probability 0, since it is continuous. Then $\mathbb{P}(\exists n \geq N |Z_n^{(1)}| > \epsilon) = 0$ and hence $\lim_{N\to\infty} \mathbb{P}(\exists n \geq N |Z_n^{(1)}| > \epsilon) = 0$

4.2:

By similar arguments we obtain that also $Z_n^{(2)}$ converges in probability but not a.s. Define $A_k = \{Z_k^{(2)} > \epsilon\}$. Then $\sum_{k\geq 1} \mathbb{P}(A_k) = \infty$ and by Borel-Cantelli we have that a.s. $Z_k^{(2)} > \epsilon$ infinitely oftern, hence it does not converge.