Exam Advanced Probability TW 3560
13th April 2015, 9:00-12:00

- The exam is a closed book exam. You may use a simple non-graphical calculator.
- In the **first part** there are 5 questions. Every correct answer gives 1 point, you can reach maximally 5 points.
- For the **second part** the points are distributed as follows:

	Exercise 1	Exercise 2	Exercise 3	Exercise 4
Points	2-2-1-2	1-3-2-2	1-1-1-1-1-2-3	1-3-2-2-2

- The total number of points is 40. The grade is calculated in the following way: grade=$\frac{1}{4}$* number of reached points.

Part I
Indicate if the following statements are true or false and explain why.

a) The random variables X, Y, Z are independent iff they are pairwise independent.

b) If $\sum_n \mathbb{P}(A_n) < \infty$ and $\{A_n\}_n$ are independent then $\mathbb{P}(\{A_n i.o.\}) = 1$.

c) If $\varphi_X(t) = \varphi_Y(t)$ for all t then the laws of X and Y are the same.

d) If $\Omega = \mathbb{R}$ then $\mathscr{F} = \mathscr{P}(\Omega)$ with some appropriate $\mathbb{P}$ defines a probability triple.

e) Fubini's theorem ensures that we can exchange the order of integration if the product integral of the absolute value of a function is finite.

Part II

Exercise 1: Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables with $\mathbb{E}(X_n) = 0$ and $\mathbb{E}(X_n^2) < \infty$ for all $n \in \mathbb{N}$. Let also $\{S_n\}_{n\geq 1}$ be the process defined as $S_0 = 0, S_n = \sum_{j=1}^n X_j$, $n \geq 1$. Assume first that

$$Cov(X_j, X_k) = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

a) Compute $\mathbb{V}(S_n)$ for all $n \in \mathbb{N}$ and $Cov(S_n, S_m)$ for $n \geq m$.

b) Does there exist a number $c \in \mathbb{R}$ such $\frac{S_n}{n} \underset{n\to\infty}{\longrightarrow} c$ in probability? Prove or disprove.

Assume now

$$Cov(X_j, X_k) = 1 \text{ for all } j, k \geq 1.$$

c) Compute $\mathbb{V}(S_n)$ for all $n \in \mathbb{N}$.

d) Does there exist a number $c \in \mathbb{R}$ such $\frac{S_n}{n} \underset{n\to\infty}{\longrightarrow} c$ in probability? Prove or disprove. (Hint: Show first that almost surely $X_j = X_k$ for any $j, k \geq 1$.)

Exercise 2: On a classical roulette game with 38 numbers (including the 0 and the 00 which are neither black nor red), a player bets uniquely on red, 361 times in a row. At each turn, he bets exactly one Euro (he therefore wins one Euro if red comes out and loses one Euro if this is not the case). Assuming that the roulette wheel is balanced and that the turns are independent from each other, calculate:

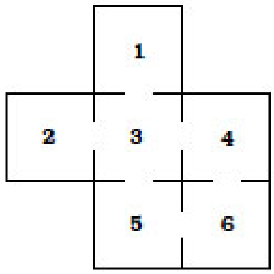
a) the probability to win in a single game.

a) the average player's fortune at the end of the 361 games;

b) the probability that he has actually won some money.

c) If you are approximating, what is the maximal error you do?

Exercise 3: A mouse runs through the maze shown below. At each step it leaves the room, it does it by choosing at random one of the doors out of the room.



a) Give the transition matrix P for this chain.

b) Draw the graphical representation of the chain.

c) Determine whether the chain is irreducible.

d) Show the chain is not aperiodic.
(Hint: You can use that: If a i is aperiodic then there exists a $k \geq 1$ such that for all $n \geq k$, $\mathbb{P}(X_n = i | X_0 = i) > 0$.)

e) Find the stationary distribution(s).

f) Suppose a piece of cheese is placed on a deadly trap in room 5. The mouse starts in room 1. Find the expected number of steps before reaching room 5 for the first time, starting in room 1.

g) Find the expected time to return to room 1 starting from room 5.

Exercise 4): Let $\{Z_n\}_n$ denote random variables whose probability mass function is defined as $\mathbb{P}(Z_n = 0) = \frac{1}{n}$ and $\mathbb{P}(Z_n = 1) = 1 - \frac{1}{n}$. Furthermore let $Y_n \sim N(\frac{1}{n}, 1)$. We assume that the sequences $(Z_n)_n$ and $(Y_n)_n$ are independent. We define a new sequence $(X_n)_n$ of random variables by $X_n := Z_n Y_n$.

a) What is the law of X_n?

b) Compute, $\mathbb{E}(X_n), \mathbb{E}(X_n^2)$ and $\mathbb{V}(X_n)$.

c) Compute the characteristic function $\varphi_{X_n}(t)$ for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$.

d) Compute $\lim_{n\to\infty} \varphi_{X_n}(t)$.

e) Determine with explanation whether or not $\mathscr{L}(X_n) \underset{n\to\infty}{\longrightarrow} \nu$. Do also the corresponding moments converge?