Solutions - Advanced Probability

TW 3560

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Part I

a) No, see definition 3.2.1.

b) No, see Borel-Cantelli, it has to be $\mathbb{P}(A_n i.o.) = 0$.

c) Yes, this is corrolary 11.1.7.

d) No, this can never define a proper probability triple.

e) Yes.

Part II

Exercise 1a): The X_i are uncorrelated hence $\mathbb{V}(S_n) = \sum_{j=1}^{n} \mathbb{V}(X_j) = n$. For $n \geq m$, we have

$$\begin{aligned} Cov(S_n, S_m) &= \mathbb{E}(S_n S_m) = \mathbb{E}(S_m^2) + \mathbb{E}((S_n - S_m)S_m) \\ &= \mathbb{E}(S_m^2) + \mathbb{E}(S_n - S_m)\mathbb{E}(S_m) = \mathbb{E}(S_m^2) = m \end{aligned}$$

1b): Yes, for any $\epsilon > 0$

$$\mathbb{P}\left(\left|\frac{S_n}{n}\right| > \epsilon\right) = \mathbb{P}(S_n^2 > \epsilon^2 n^2) \leq \frac{\mathbb{E}(S_n^2)}{\epsilon^2 n^2} = \frac{1}{\epsilon^2 n} \underset{n\to\infty}{\longrightarrow} 0.$$

1c):

$$\mathbb{V}(S_n) = \sum_{j=1}^{n} \mathbb{V}(X_j) + \sum_{j\neq k} Cov(X_j, X_k) = n + n(n-1) = n^2$$

1d): For any $j \geq 1$ and $k \geq 1$ we have $\mathbb{E}(X_j^2) = \mathbb{E}(X_k^2) = \mathbb{E}(X_j X_k) = 1$ implying $\mathbb{E}((X_j - X_k)^2) = 0$ hence $X_j = X_k$ almost surely. Therefore $\frac{S_n}{n} = X_1$ so it does not converge in probability almost surely.

Exercise 2): Let X_i model the outcome of the game i. $X_i = 1$ means that we won, $X_i = -1$ is modelling loss. The sequence of gains of the player is the i.i.d. sequence $(X_1, ..., X_{361})$ with $\mathbb{P}(X_i = 1) = \frac{18}{38}$ and $\mathbb{P}(X_i = -1) = \frac{20}{38}$.

a): $\mathbb{P}(X_1 = 1) = \frac{18}{38}$

b):

$$\mu = \mathbb{E}(X_1) = \frac{18-20}{38} = -\frac{1}{19}$$

and

$$\mathbb{V}(X_1) = 1^2\frac{18}{38} + (-1)^2\frac{20}{38} - \frac{1}{19\cdot 19} = 1 - \frac{1}{361}$$

Let $S_n = X_1 + ... + X_n$ denote the gain after n games. Then $\mathbb{E}(S_{361}) = 361\mu = -19$ Euros.

c): By the central limit theorem:

$$\mathbb{P}(|S_{361}| > 0) = \mathbb{P}\left(\frac{S_{361} - 361\mu}{\sqrt{361\mathbb{V}(X_1)}} > -\frac{\sqrt{361}\mu}{\sqrt{\mathbb{V}(X_1)}}\right) \approx \mathbb{P}\left(Z > -\frac{\sqrt{361}\mu}{\sqrt{\mathbb{V}(X_1)}}\right)$$
$$\approx \mathbb{P}(Z > 1) \approx 0.15$$

where $Z \sim N(0,1)$.

d): We calculate

$$\mathbb{E}(|X_i - \mu|^3) = \left(\frac{18}{19}\right)^3\frac{18}{38} + \left(\frac{20}{19}\right)^3\frac{20}{38} \approx 1.02$$

and the Berry-Esséen bound gives $\frac{3\cdot 1.02}{\sqrt{361*(0.99)^3}} \approx 0.16$.

Exercise 3 a): The transition matrix is

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

c): The chain is irreducible, it is possible to reach every room from every other room.

d): The chain is not aperiodic. Since it is irreducible we have to choose aperiodicity for one state. A state i is said to be aperiodic if there exists n such that for all $m \geq n$, $\mathbb{P}(X_m = i | X_0 = i) > 0$. State i is not aperiodic because it is only possible to return to it in even steps. For odd n the probability is 0.

e): We have to find $\pi^T = (\pi_1, ..., \pi_6)$ such that $\pi^T P = \pi^T$, and $\pi_1 + ... + \pi_6 = 1$. We get

$$\pi^T = \left(\frac{1}{12}, \frac{1}{12}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$$

f): Let $\tau = \inf\{n \geq 0 : X_n = 5\}$, we look for $\Psi(i) = \mathbb{E}(\tau | X_0 = i)$. We write down the system of equations:

$$\begin{aligned}
\Psi(1) &= 1 + \Psi(3) \\
\Psi(2) &= 1 + \Psi(3) \\
\Psi(3) &= 1 + \frac{1}{4}\Psi(1) + \frac{1}{4}\Psi(2) + \frac{1}{4}\Psi(4) + \frac{1}{4}\Psi(5) \\
\Psi(4) &= 1 + \frac{1}{2}\Psi(6) + \frac{1}{2}\Psi(3) \\
\Psi(5) &= 0 \\
\Psi(6) &= 1 + \frac{1}{2}\Psi(5) + \frac{1}{2}\Psi(4)
\end{aligned}$$

We see that $\Psi(1) = \Psi(2)$, furthe we solve the system and find $\Psi(3) = 6$ and hence $\Psi(1) = 7$.

g): Let $\tau = \inf\{n \geq 0 : X_n = 1\}$, we look for $\Psi(i) = \mathbb{E}(\tau | X_0 = i)$. We write down the system of equations:

$$\begin{aligned}
\Psi(1) &= 0 \\
\Psi(2) &= 1 + \Psi(3) \\
\Psi(3) &= 1 + \frac{1}{4}\Psi(1) + \frac{1}{4}\Psi(2) + \frac{1}{4}\Psi(4) + \frac{1}{4}\Psi(5) \\
\Psi(4) &= 1 + \frac{1}{2}\Psi(6) + \frac{1}{2}\Psi(3) \\
\Psi(5) &= 1 + \frac{1}{2}\Psi(3) + \frac{1}{2}\Psi(6) \\
\Psi(6) &= 1 + \frac{1}{2}\Psi(5) + \frac{1}{2}\Psi(4)
\end{aligned}$$

We see that $\Psi(2) = 12$, $\Psi(3) = 11$, $\Psi(4) = \Psi(5) = 14$ and $\Psi(6) = 15$. The answer is $\Psi(5) = 14$.

Exercise 4 a): Let B be a Borel measurable set.

$$\begin{aligned}
\mathbb{P}(X_n \in B) &= \mathbb{P}(X_n \in B | Z_n = 0)\mathbb{P}(Z_n = 0) + \mathbb{P}(X_n \in B | Z_n = 1)\mathbb{P}(Z_n = 1) \\
&= \mathbb{P}(0 \in B)\frac{1}{n} + (1 - \frac{1}{n})\mu(B)
\end{aligned}$$

hence $\mathscr{L}(X_n) = \frac{1}{n}\delta_0 + (1 - \frac{1}{n})\mu$ where $\mu = N(\frac{1}{n}, 1)$.

b):

$$\mathbb{E}(X_n) = \frac{1}{n}\int_{-\infty}^{\infty} t\delta_0(dt) + (1 - \frac{1}{n})\int_{-\infty}^{\infty} t\mu(dt) = (1 - \frac{1}{n})\frac{1}{n}$$

$$\mathbb{E}(X_n^2) = \frac{1}{n}\int_{-\infty}^{\infty} t^2\delta_0(dt) + (1 - \frac{1}{n})\int_{-\infty}^{\infty} t^2\mu(dt) = (1 - \frac{1}{n})(1 + \frac{1}{n^2})$$

and hence $\mathbb{V}(X_n) = 1 - \frac{1}{n}$.

c):

$$\begin{aligned}\varphi_{X_n}(t) &= \frac{1}{n}\int_{-\infty}^{\infty} e^{itx}\delta_0(dx) + (1-\frac{1}{n})\int_{-\infty}^{\infty} e^{itx}\frac{e^{-(x-1)^2/2}}{\sqrt{2\pi}}dx \\ &= \frac{1}{n} + (1-\frac{1}{n})e^{i\frac{1}{n}-\frac{1}{2}t^2}\end{aligned}$$

d): $\lim_{n\to\infty}\varphi_{X_n}(t) = e^{-\frac{1}{2}t^2}$.

e): Yes by continuity theorem the sequence converges weakly towards $\nu = N(0,1)$. The moments also converge.