Retake: Continuous Optimisation 2016

Monday 16th January 2017

Hints at the end of the paper. Workings must be shown. Good Luck!

1. Consider the problem $\min_x\{x^2 : x \geq 1\}$. For a parameter $\rho > 0$, this problem can be approximated by the unconstrained optimisation problem [3 points]

$$\begin{aligned} \min_x \quad & x^2 - \rho \ln(x-1) \\ \text{s. t.} \quad & x > 1. \end{aligned} \tag{A}$$

Find the optimal solutions to (A) as a function of $\rho > 0$, and find the limit of these optimal solutions as $\rho \to 0^+$.

2. Consider a closed nonempty set $C \subseteq \mathbb{R}^n$ and a function $f : \mathbb{R}^n \to \mathbb{R}$ defined to be the distance to the set for some given norm, i.e. [3 points]

$$f(\mathbf{x}) = \min_{\mathbf{y}}\{\|\mathbf{x}-\mathbf{y}\| : \mathbf{y} \in C\}.$$

Prove that if C is a convex set then f is a convex function.
[You may assume that the minimum defining f is attained.]

3. For a fixed parameter $\alpha \in \mathbb{R}$, consider the function $f_\alpha(\mathbf{x}) = \exp(x_1+x_2)+\alpha x_1^2+x_2^4$.

(a) For what values of the parameter $\alpha \in \mathbb{R}$ is f_α a convex function? [3 points]

From now on consider having $\alpha = 1$.

? (b) By considering the function at $\mathbf{x} = \mathbf{0} = \begin{pmatrix}0\\0\end{pmatrix}$, show that $f_1(\mathbf{y}) \geq 1 + y_1 + y_2$ for all $\mathbf{y} \in \mathbb{R}^2$. [2 points]

(c) Give the direction of steepest descent of f_1 at $\mathbf{x} = \mathbf{0}$. [1 point]

(d) Give the Newton direction of f_1 at $\mathbf{x} = \mathbf{0}$. [2 points]

[These directions do not need to be normalised.]

4. Consider the problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & 4x_1 + x_2^2 \\ \text{s. t.} \quad & x_2 \geq x_1^2 \\ & \mathbf{x} \in \mathbb{R}^2. \end{aligned} \tag{B}$$

(a) Show that problem (B) is a convex problem. [2 points]

(b) Does Slater's condition hold for problem (B)? (You must justify your answer.) [1 point]

(c) Find the KKT point(s) for problem (B). [3 points]

(d) What is the global minimiser for problem (B), and prove that this minimiser is a local minimiser of order 2. [3 points] -1

(e) Formulate and solve the Langrangian Dual problem to problem (B). [4 points] -2

5. For vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, consider the set $\mathcal{K} = \operatorname{conic}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$. Show that $\mathcal{K}$ is a proper cone if and only if $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are linearly independent vectors. [4 points]

6. Consider the following two optimisation problems, where $\mathcal{K} \subseteq \mathbb{R}^n$ is a proper cone, $\mathcal{K}^* \subseteq \mathbb{R}^n$ is its dual cone, $\mathbf{c}, \mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^m$: [2 points]

$$\begin{aligned} \min_{\mathbf{x}} \quad & \langle \mathbf{c}, \mathbf{x} \rangle \\ \text{s. t.} \quad & \langle \mathbf{a}_i, \mathbf{x} \rangle \geq b_i \quad \text{for all } i = 1, \ldots, m \\ & \mathbf{x} \in \mathcal{K} \end{aligned} \tag{C}$$

$$\begin{aligned} \max_{\mathbf{y}} \quad & \mathbf{b}^\mathsf{T}\mathbf{y} \\ \text{s. t.} \quad & \mathbf{c} - \sum_{i=1}^{m} y_i \mathbf{a}_i \in \mathcal{K}^* \\ & \mathbf{y} \in \mathbb{R}^m_+ \end{aligned} \tag{D}$$

Show that weak duality holds between these problems.

7. We will consider bounds to the optimal value of the following problem:

$$\begin{aligned} \min_{\mathbf{x}} \quad & 4x_1x_2 + 3x_2^2 - x_2 \\ \text{s.t.} \quad & 5x_1^2 - 2x_1x_2 - 3x_2^2 + 5x_1 + 12x_2 = 0 \\ & \mathbf{x} \in \mathbb{R}^2. \end{aligned} \tag{E}$$

(a) Give a finite upper bound on the optimal value of problem (E). [1 point]

(b) Formulate a positive semidefinite optimisation problem whose solution would give a lower bound on the optimal value of problem (E). [2 points]

8. (Automatic additional points) [4 points]

Question:	1	2	3	4	5	6	7	8	Total
Points:	3	3	8	13	4	2	3	4	40

A copy of the lecture-sheets may be used during the examination. You may use any results from the lecture slides in your answers (Lemmas, Theorems, Corollaries, Exercises, etc.), however you should reference the result.

Hints:

1. $\begin{pmatrix} a & b \\ b & c \end{pmatrix}^{-1} = \dfrac{1}{ac - b^2} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix}$

2. *A norm $\|\bullet\|$ on $\mathbb{R}^n$ has the following properties:*
   (a) $\|\lambda \mathbf{x}\| = |\lambda| \, \|\mathbf{x}\|$ *for all* $\lambda \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^n$;
   (b) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ *for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$;
   (c) $\|\mathbf{x}\| > 0$ *for all* $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$.