Technische Universiteit Delft — Mekelweg 4
Faculteit Elektrotechniek, Wiskunde en Informatica — 2628 CD Delft

Exam Statistical Inference (WI4455)
January 26, 2017, 9.00–12.00

Using books or notes is not allowed at the exam.
Unless stated diferently, always add an explanation to your answer.

1. Let $X_1, \ldots, X_n$ be a random sample from the distribution with density

$$f(x \mid \theta) = \frac{2x}{\theta^2} \mathbf{1}_{(0,\theta)}(x) \qquad (*)$$

with respect to Lebesgue measure.

   (a) Derive an unbiased estimator for θ. Does this estimator respect the likelihood principle?
   (b) Show that $T = \max(X_1, \ldots, X_n)$ is a sufficient statistic.
   (c) Show that the density of T if given by $2nt^{2n-1}\theta^{-2n}$ for $0 < t < \theta$.
   (d) Show directly that T is complete.
   (e) Is T minimal sufficient for θ?
   (f) Now suppose $n = 1$. Assume $X \mid \Theta = \theta$ has density as in $(*)$. Let Θ have a uniform distribution on $(0, c)$, where c a known (fixed) positive number. Find an expression for the posterior density of Θ.
   (g) Compute the posterior mean in the setting of exercise (f).

2. Let X be a random variable that takes values in $\{-1, 0, 1\}$. We want to test

$$H_0 : \Pr(X = -1) = \Pr(X = 0) = \Pr(X = 1) = 1/3$$

versus

$$H_1 : \Pr(X = -1) = \Pr(X = 1) = 1/4 \qquad \Pr(X = 0) = 1/2.$$

In this exercise we will use various ways to decide upon either H_0 or H_1.

   (a) Show that the uniformly most powerful test of size $\alpha = 1/3$ based on X leads to rejecting the null hypothesis when $X = 0$.
   (b) Calculate the power of this test.
   (c) We now take a decision theoretic point of view. We assume Θ is a random variable that takes values in $\Omega = \{0, 1\}$ and consider the random variable X that has the following probability mass function, conditional on Θ,

$$\Pr(X = -1 \mid \Theta = 0) = \Pr(X = 0 \mid \Theta = 0) = \Pr(X = 1 \mid \Theta = 0) = 1/3$$

$$\Pr(X = -1 \mid \Theta = 1) = \Pr(X = 1 \mid \Theta = 1) = 1/4 \qquad \Pr(X = 0 \mid \Theta = 1) = 1/2.$$

   We wish to find the optimal decision rule $d : \mathcal{X} \to \Omega$, where $\mathcal{X} = \{-1, 0, 1\}$. We consider the loss function $L(\theta, d)$ with $L(0,0) = L(1,1) = 0$, $L(1,0) = 1$ and $L(0,1) = 3$.

i. There are eight non-randomised decision rules in this problem. These are given in the following table:

	$x=-1$	$x=0$	$x=1$	$R(0,d)$	$R(1,d)$
$d_1(x)$	0	0	0	0	1
$d_2(x)$	0	0	1		
$d_3(x)$	0	1	0		
$d_4(x)$	0	1	1		
$d_5(x)$	1	0	0		
$d_6(x)$	1	0	1		
$d_7(x)$	1	1	0		
$d_8(x)$	1	1	1		

Hence d_1 always decides $\theta = 0$ (irrespective the value of x), d_2 decides $\theta = 1$ only if $x = 1$, etc.
Complete the table.

ii. Find the minimax rule.

iii. Find the Bayes rule if a priori $\Pr(\Theta = 0) = 2/3$.

(d) Suppose the prior probability of H_0 equals 2/3 and $x = 0$ is observed. What is the posterior probability of H_0?

3. At a critical stage in the development of a new aeroplane, a decision must be taken to continue or to abandon the project. The financial viability of the project can be measured by a parameter $\theta \in (0,1)$, the project being profitable if $\theta > 1/2$. Data x provide information about θ. If $\theta < 1/2$, the cost to the taxpayer of continuing the project is $1/2 - \theta$ (in units of 5 billion), whereas if $\theta > 1/2$ it is zero (since the project will be privatised if profitable). If $\theta > 1/2$ the cost of abandoning the project is $\theta - 1/2$ (due to contractual arrangements for purchasing the aeroplane from the French), whereas if $\theta < 1/2$ it is zero.

   Denote the decision to continu the project by d_1. Denote the decision to abandon the project by d_2.

   Show that the optimal Bayesian decision is to continue the project if the posterior mean of θ is greater than 1/2.

4. If $X \sim Pois(\theta)$ with $\theta > 0$ ($P'_\theta(X = x) = e^{-\theta}\theta^x/x!$), then X is a complete sufficient statistic (you don't need to prove this). Find the uniformly minimum variance unbiased (UMVU) estimator for $e^{-3\theta}$.

   [Recall that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.]

5. State what is meant by the conditionality principle in statistical inference.