WI4201 Written Exam Resit
April 13th, 2017

This examination consists of the following **5 questions**. *Read the questions* **carefully** *before answering. With each question ten credit points can be gained. This exam is an closed book exam. You are allowed a sufficiently simple calculator. You are* **not** *allowed to use any book or notes.*

Question 1 (10 pnts. - 2 pnts. per subquestion)
Answer the following questions

1. give an example of a 3-by-3 non-symmetric non-diagonal matrix A that is row diagonally dominant;

2. an n-by-n matrix is positive semidefinite if and only if for all n-vectors $\mathbf{u} \neq \mathbf{0}$ holds that $\mathbf{u}^T A\mathbf{u} \geq 0$. Let A be the n-by-n matrix with all entries equal to one. Show that A can be written as $A = \mathbf{e}^T\mathbf{e}$ where $\mathbf{e}$ is the row vector $\mathbf{e} = (1, 1, \ldots, 1)$ and show that A is positive semidefinite;

3. show using an example with 2-by-2 matrices that the functional $\mathcal{F}$ defined on the set of n-by-n matrices A as $\mathcal{F}(A) = \max_{1 \leq i,j \leq n}(A_{ij})$ is **not** submultiplicative;

4. give a distinct advantage of the GMRES method over the BiCG method for solving a linear system with a non-symmetric coefficient matrix;

5. show that the condition number of a square matrix for a submultiplicative norm is a number between 1 and $+\infty$;

Question 2 (10 pnts. - 2 pnt. per subquestion)

Given a positive real number $\epsilon > 0$, given the interval $0 \leq x \leq 1$, and given the known function $f(x)$ on this interval, we consider the following convection-diffusion equation for the unknown concentration $u(x)$

$$-\underbrace{\epsilon \frac{d^2u}{dx^2}}_{diffusion} + \underbrace{\frac{du}{dx}}_{convection} = 0 \text{ for } 0 < x < 1\,, \tag{1}$$

supplied with the ~~homogeneous~~ Dirichlet boundary conditions at both end points

$$u(x=0) = 0 \text{ and } u(x=1) = 1\,. \tag{2}$$

We will consider the finite difference approximation of this problem using a uniform intervals with N elements and mesh width $h = 1/N$. The second order central finite difference scheme is used to discretize the diffusion term. The first order upwind scheme is used to discretize the convection term. The grid nodes are numbered as $x_i = (i-1)h$ where $1 \leq i \leq N+1$. Using this notation, the left and right most grid x_1 and $x_{N+1} = 1$ correspond to the boundary points $x = 0$ and $x = 1$, respectively. Let f_i and u_i denote the values of the given function $f(x)$ and the unknown function $u(x)$ in the grid point x_i, respectively. These values are collected in the vectors $\mathbf{f}$ and $\mathbf{u}$. The discretization of the differential equation supplied with boundary conditions then results in the linear system of equations

$$A\mathbf{u} = \mathbf{f}\,. \tag{3}$$

1. give the finite difference discretization of Equation (1) for an interior grid point x_i. This discretization should include both the diffusive and the convective term;
2. discuss the finite difference treatment of the Dirichlet boundary condition in $x = 0$;
3. consider the special case in which $N = 3$ and give for this case the matrix A of the linear system (3);
4. show that for all mesh widths h the coefficient matrix A is an M-matrix;
5. the incomplete LU (ILU) method applied either as basic iterative method or as preconditioner to this linear system converges in a single iteration. Explain why this is the case;

Question 3 (10 pnts. - 2 pnt. per subquestion)
Assume that

$$[A] = \frac{1}{h^2}\begin{bmatrix} -1 & 2 & -1 \end{bmatrix}$$

is the stencil of the 1D Laplacian on a uniform mesh. Assume that the $(N+1)\times(N+1)$ coefficient matrix A results from the assembly of the matrix on a mesh with mesh width h with homogeneous Dirichlet boundary conditions at both end points.

1. Assume given the linear system $A\mathbf{u} = \mathbf{f}$. Assume a splitting of this coefficient matrix of the form $A = M - N$ where M is non-singular and assume that a basic iterative solution method for the linear system is derived from this splitting. Derive a recursion formula for the iterands $\mathbf{u}^k$. Derive a recursion formula for the error vector $\mathbf{e}^k$.

2. assume that the asymptotic rate of convergence of the Jacobi method applied to A is given by $\rho_{JAC} = \cos(h\,\pi)$. Give the asymptotic rate of convergence of the Gauss-Seidel method applied to A.

3. Assume that the eigenvectors $\mathbf{v}^k$ of $A \in \mathbb{R}^{(N+1)\times(N+1)}$ are given by

$$v_i^k = \sin(k\,\pi\,x_i) = \sin(k\,\pi\,(i-1)\,h) \text{ for } 1 \le i \le N+1\,.$$

Derive an expression for the corresponding eigenvalue λ_k as a function of the meshwidth h by computing the action of A on $\mathbf{v}^k$. Neglect here the treatment of the boundary conditions.

4. Assume a Red-Black ordering of the grid nodes. Give the sparsity pattern of the matrix A is this ordering. Show that after one Red-Black Gauss-Seidel sweep the residual vector $\mathbf{r}^k$ is zero in the components corresponding to the Red nodes.

5. Consider that the fine mesh with mesh width h in combined with a coarse mesh with mesh width $H = 2h$. Describe a two-grid iterative procedure in which a Jacobi sweep on the fine mesh is combined with the defect correction scheme using the coarse mesh in a $V(1,0)$-cycle;

Question 4 (10 pnts. - 2.5 pnt. per subquestion)

In this exercise A is an n-by-n SPD matrix.

1. Assume $\mathbf{f}$ to be a given n-vector. Give an outline of the Cholesky decomposition method to solve the linear system $A\,\mathbf{u} = \mathbf{f}$.

2. Give the definition of partial pivoting. Show that it is not necessary to use partial pivoting for the Cholesky method applied to A SPD.

3. Given the linear system $A\,\mathbf{u} = \mathbf{f}$ with A an n-by-n matrix and the perturbed system

$$A\,(\mathbf{u} + \Delta\mathbf{u}) = \mathbf{f} + \Delta\mathbf{f}\,.$$

   Derive an upper bound for the quantity $\|\Delta\mathbf{u}\|/\|\mathbf{u}\|$ where $\|\cdot\|$ is an arbitrary vector norm, that has the submultiplicative property.

4. Suppose that A is a tridiagonal matrix, where the elements $A(i,i-1)$, $A(i,i)$ and $A(i,i+1)$ are non-zero. Give the non-zero pattern of the Cholesky factors. Give the Cholesky decompostion algorithm for this matrix.

Question 5 (10 pnts. - 2 pnt. per subquestion)

Consider A to be a square non-singular matrix.

1. Explain why the Conjugate Gradient algorithm can be applied to $A^T A\,\mathbf{u} = A^T\mathbf{f}$ to approximate the solution the linear system $\mathbf{u}$ of $A\,\mathbf{u} = \mathbf{f}$.
2. Give advantages and disadvantages of this approach.
3. Suppose that A is given by
$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$
How many iterations are needed to solve $A\,\mathbf{u} = \mathbf{f}$ with the method given in (1)? Motivate your answer.
4. Show that the solution $\begin{pmatrix} \mathbf{y} \\ \mathbf{u} \end{pmatrix}$ of the augmented system
$$\begin{pmatrix} I & A \\ A^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix}$$
is such that $\mathbf{u}$ satisfies $A^T A\,\mathbf{u} = A^T\mathbf{f}$. State whether or not the matrix $\begin{pmatrix} I & A \\ A^T & 0 \end{pmatrix}$ is SPD or not, and motivate your answer.
5. Assume given the matrix $\widehat{A} = \begin{pmatrix} I & A \\ A^T & 0 \end{pmatrix}$ and the vectors $\widehat{\mathbf{f}} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix}$ and $\widehat{\mathbf{u}} = \begin{pmatrix} \mathbf{y} \\ \mathbf{u} \end{pmatrix}$. Show that the Krylov subspace $K^{2k}(\widehat{A}, \widehat{\mathbf{f}})$ is given by
$$K^{2k}(\widehat{A}, \widehat{\mathbf{f}}) = \text{span}\left\{ \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} AA^T\mathbf{f} \\ \mathbf{0} \end{pmatrix}, \ldots, \begin{pmatrix} (AA^T)^{k-1}\mathbf{f} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ A^T\mathbf{f} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ A^TAA^T\mathbf{f} \end{pmatrix}, \ldots, \begin{pmatrix} \mathbf{0} \\ (A^TA)^{k-1}A^T\mathbf{f} \end{pmatrix} \right\}.$$