TUDelft

Delft University of Technology

Martingales and Brownian Motion
Tuesday 18 April, 2017, 13.30-16.30

Five exercises. They all have equal payoff.

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1 Consider a sequence of independent random variables X_n for $n = 1, 2, \ldots$ taking the values zero or one, with $\mathbf{P}(X_n = 1) = a_n$.

A Prove that $X_n \to 0$ in probability if and only if $a_n \to 0$.

B Prove that $X_n \to 0$ almost surely if and only if $\sum a_n < \infty$.

C Under what conditions on a_n does X_n converge to zero in the mean?

2 Let Y be an integrable random variable which takes the integer values $0, 1, 2, \ldots$ and $\mathbf{E}[Y] = m > 0$. Let $Y_{i,j}$ be a set of independent random variables which are all distributed as Y. The indices i, j both run through $0, 1, 2, \ldots$. The filtration $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ consists of $\mathcal{F}_n$ which are generated by $Y_{i,j}$ with $i \leq n$.

Define $X_0 = 1$ and by induction

$$X_{n+1} = \sum_{k=1}^{X_n} Y_{n,k}$$

A Prove that $M_n = X_n/m^n$ is a martingale.

B Prove that M_n converges almost surely.

C Prove that $X_n \to 0$ almost surely if $m < 1$.

3 Let M_n be a non-negative submartingale and let $\lambda > 0$. Define $\tau = \min\{k\colon M_k \geq \lambda\}$.

A. Prove that τ is a stopping time.

B. Prove that $\lambda \leq \mathbf{E}[M_n \mid \{\tau \leq n\}]$.

C. Prove Doob's maximal inequality

$$\mathbf{P}(M_n^* \geq \lambda) \leq \frac{1}{\lambda}\mathbf{E}[M_n]$$

4 Let W_t be the standard Brownian Motion and let $\mathcal{F}_t$ be the natural filtration.

A Prove that $V_t = \frac{1}{\sqrt{c}} W_{ct}$ is the standard Brownian Motion for any $c > 0$.

B Prove that $W_t^2 - t$ is a martingale.

C Define $\tau = \inf\{t \colon W_t \geq 2 \text{ or } W_t \leq -2\}$. Assuming that this is an integrable stopping time (no need to verify this), compute $\mathbf{E}[\tau]$.

5 Again W_t is the standard Brownian Motion.

A W_t^4 is an Itô process and so $dW_t^4 = a(t, W_t)dt + b(t, W_t)dW_t$. Determine $a(t, W_t)$ and $b(t, W_t)$ for this process.

B Compute $\mathbf{E}\left[W_T^4\right]$ by using that

$$\mathbf{E}[W_T^4] = \mathbf{E}\left[\int_0^T dW_t^4\right] = \mathbf{E}\left[\int_0^T a(t, W_t)dt\right] + \mathbf{E}\left[\int_0^T b(t, W_t)dW_t\right]$$

That is it. Check your answers once more and hand in your form.