Martingales and Brownian Motion
Tuesday 31 Jan, 2017, 13.30-16.30

Five exercises. They all have equal payoff.

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1 Let M_n be a non-negative martingale which starts at $M_0 = 1$ for $n = 0, 1, 2, \ldots$.

A Does M_n converge in the mean? If it does, then state the appropriate theorem and show that the conditions of the theorem are satisfied. If it does not, then produce a counterexample.

B Does M_n converge almost surely? If it does, then state the appropriate theorem and show that the conditions of the theorem are satisfied. If it does not, then produce a counterexample.

2 Consider the simple random walk $S_n = X_1 + \cdots + X_n$ for i.i.d. random variables X_i which are equiprobably $+1$ or -1. As usual, $\mathcal{F}_n$ is the σ-algebra generated by $X_1, \cdots, X_n$. The random walk starts at $S_0 = 0$.

A Compute $\mathbf{E}[S_n^4 \mid \mathcal{F}_{n-1}]$. Conclude that S_n^4 is a submartingale.

B According to Doob's decomposition $S_n^4 = M_n + A_n$ for a martingale M_n and a predictable process A_n which both start at zero. Determine M_n and A_n for $n = 1, 2, 3$.

C Define $\tau = \inf\{n \colon S_n < \frac{n}{2}\}$. Prove that τ is a finite stopping time.

3 Let $X_1, X_2, \ldots$ be independent random variables such that

$$X_n = \begin{cases} n^2 - 1 & \text{with probability } n^{-2} \\ -1 & \text{with probability } 1 - n^{-2} \end{cases}$$

Define $M_n = X_1 + \cdots + X_n$ and let $\mathcal{F}_n$ be the usual filtration. The process starts at $M_1 = 0$ since $X_1 = 0$.

A. Verify that M_n is a martingale.

B. Verify that $M_n \to -\infty$ almost surely.

C. Define $\tau = \inf\{n > 1 \colon X_n \neq -1\}$. Verify that $M_\tau > 0$ if $\tau < \infty$. Why does this not contradict Doob's optional sampling theorem?

4 Let W_t be the standard Brownian Motion and let $\mathcal{F}_t$ be the natural filtration. Recall that $\mathbf{E}[Z^4] = 3$ and that $\mathbf{E}[e^{\lambda Z}] = e^{\lambda^2/2}$ if Z is a standard normal random variable.

A. Compute $\mathbf{E}[(W_s^2 - s)(W_t^2 - t)]$ for $0 < s < t$.

B. Compute $\mathbf{E}[e^{W_t} \mid \mathcal{F}_s]$ for $0 < s < t$.

C. Define $V_t = tW_{1/t}$ for $t > 0$. Verify that the increments $V_t - V_s$ for $0 < s < t$ are normally distributed with mean zero and variance $t - s$.

5 Again W_t is the standard Brownian Motion.

A. Compute the differential of the Itô process $X_t = W_t^3 - 3tW_t$. Conclude that it is a continuous martingale.

B. Use the Itô isometry to compute the variance of X_t.

C. Solve the following SDE:

$$dY_t = -Y_t dW_t$$

That is it. Check your answers once more and hand in your form.