Exam Martingales, Brownian Motion and Stochastic Processes WI 4430.

26 january 2016, 13:30-16:30.

No books or notes allowed.

Responsible of the course: Prof. F. Redig

Second reader exam: Dr. Ludolf Meester

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a) The exam consists of two theory questions, each on 10 points, followed by exercises. The exercises consist of 10 small questions each on 2 points.

b) The end score is computed as explained on the blackboard page. Course grade is the final exam grade f or $0.6f + 0.4h$ (with h average homework grade), whichever is larger, provided $f \geq 5$.

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1 Theory Questions.

1) a) Give the definition of a martingale.

   b) Give the definition of a stopping time. Give both an example of a stopping time and an example of a time which is not a stopping time.

   c) State and prove the stopped martingale theorem, i.e., prove that a stopped martingale is still a martingale.

2) a) Give the definition of Brownian motion.

   b) Give the definition of the Brownian bridge.

   c) Prove that if $\{X(t), t \geq 0\}$ is Brownian motion, then $X(t) - tX(1)$ is a Brownian bridge

2 Exercises.

1. Let X_i, $\imath \in \mathbb{N}$, denote independent random variables for which

$$\mathbb{P}(X_i = 2) = \mathbb{P}(X_i = \tfrac{1}{2}) = \tfrac{1}{2}.$$

and let $\mathscr{F}_n$ denote the filtration $\sigma\{X_i : 1 \leq i \leq n\}$

a) Is

$$Y_n := \prod_{i=1}^{n} X_i$$

a $\mathscr{F}_n$ martingale, submartingale, supermartingale?

b) Compute the conditional expectation

$$\mathbb{E}(Y_{n-1}Y_n^2 | \mathscr{F}_{n-1})$$

c) Prove that $Y_n^{1/n}$ converges almost surely to 1. (Hint: take logarithms).

2. Let $\{W_t, t \geq 0\}$ denote Brownian motion, and let $\mathscr{F}_t$ be the corresponding filtration. Let $\mu > 0$ be given. The process $\{X_t : t \geq 0\}$ defined via

$$X_t := \mu t + W_t \tag{1}$$

is called Brownian motion with drift μ.

a) Compute the expectation $\mathbb{E}(X_t^2)$ and for $0 < s < t$, $\lambda > 0$ the expectation $\mathbb{E}(e^{\lambda X_t} X_s^2)$.

b) Prove that $M_t := (X_t - \mu t)$ and $\tilde{M}_t := (X_t - \mu t)^2 - t$ are $\mathscr{F}_t$ martingales.

c) Show that $Z_t := e^{-2\mu X_t}$ is also a $\mathscr{F}_t$ martingale.

d) Consider for $a > 0$ the stopping time

$$\tau = \inf\{t > 0 : X_t \notin [-a, a]\}$$

(you are allowed to use that this is a stopping time for which $\mathbb{E}(\tau) < \infty$). Stopping the martingale in item c), prove the formula

$$\mathbb{P}(X_\tau = a) = \frac{e^{2\mu a} - 1}{e^{2\mu a} - e^{-2\mu a}}$$

e) Using the result from item d), and using the martingale M_t from item b), compute the expectation $\mathbb{E}(\tau)$.

3. As before, we denote Brownian motion by $\{W_t, t \geq 0\}$. Using Ito's formula prove that

$$W_t^3 - 3tW_t = \int_0^t 3(W_s^2 - s)dW_s$$

4. Using the formula obtained in the previous question, Ito's isometry and the fact that $\mathbb{E}(W_t^4) = 3t^2$, prove that

$$\mathbb{E}\left((W_t^3 - 3tW_t)^2\right) = 6t^3$$