**Exam Time Series and Extreme Value Theory (WI4230) – Time Series Part**

Delft University of Technology, Department of EEMCS, 28 June 2017, 13.30 – 16.30

---

The use of electronic devices (including graphical or basic calculators) is *not* allowed. Please write your name and student number on every sheet, and keep your student card ready for inspection.

This exam consists of 2 pages with 5 questions. There are 18 parts in total, each worth 2 points.

Notation: $(Z_t) \sim \mathrm{WN}(0, \sigma^2)$ indicates that $(Z_t : t \in \mathbb{Z})$ is a white noise process with variance $\sigma^2$.

---

1. (a) What is the definition of a strictly stationary time series? What is the definition of a (weakly) stationary time series?

   (b) Is a strictly stationary time series always weakly stationary? Why/why not?

2. Consider the ARMA equation

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}, \quad t \in \mathbb{Z},$$

   where $(Z_t : t \geq 0)$ is a white noise process with variance $\sigma^2$.

   (a) Describe a (non-trivial) sufficient condition in order for a stationary solution to exist, and give a brief sketch of the proof that such a solution exists in this case.

   (b) Suppose the ARMA equation admits a stationary solution. What does it mean for such a solution to be invertible? What is a sufficient condition for invertibility?

   (c) Consider the equation

$$Y_t = \alpha_0 + \alpha_1 t + \beta_1 Y_{t-1} + \beta_2 Y_{t-2} + \gamma_0 W_t + \gamma_1 W_{t-1}, \quad t \in \mathbb{Z},$$

   with $(W_t) \sim \mathrm{WN}(0, \tau^2)$. This equation can be rephrased, through a suitable transformation of the time series $(Y_t : t \geq 0)$, as the ARMA(2,2)-equation

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}, \quad t \in \mathbb{Z},$$

   where $(Z_t) \sim \mathrm{WN}(0, \sigma^2)$. What is this transformation and what are the resulting values of $(\phi_1, \phi_2, \theta_1, \theta_2, \sigma^2)$ in terms of $(\alpha_0, \alpha_1, \beta_1, \beta_2, \gamma_0, \gamma_1, \tau^2)$?

   As a specific example, consider the ARMA equation

$$X_t = -\tfrac{3}{2} X_{t-1} + X_{t-2} + Z_t - 4 Z_{t-2}.$$

   (d) Does this equation admit a stationary solution? If so, is it causal? Invertible? Motivate your answer.

   (e) Determine the spectral density function of $(X_t)$. (Correctness of your answer is more important than complete simplification.)

3. Suppose $(X(t) : t \in \mathbb{Z})$ is a stationary solution to the MA(1) equation

$$X_t = Z_t + \theta Z_{t-1}, \quad t \in \mathbb{Z},$$

   with $|\theta| < 1$.

(a) What is the autocovariance function of $(X_t)$? (The computation is optional.)

We wish to find the best linear predictor for $X_{n+1}$ given observations $X_1, \ldots, X_n$.

(b) State, for a general stationary process, the linear prediction equations in matrix form. Show that for this example, with $n \geq 2$, these lead to the system of equations

$$\begin{aligned} \theta\phi_{n,j-1} + (1+\theta^2)\phi_{n,j} + \theta\phi_{n,j+1} &= 0, \qquad\qquad 2 \leq j \leq n-1, \\ (1+\theta^2)\phi_{n,n} + \theta\phi_{n,n-1} &= 0, \\ (1+\theta^2)\phi_{n,1} + \theta\phi_{n,2} &= \theta. \end{aligned}$$

(c) Explain how we can, in principle, determine a solution of the recursive system above. What is the general solution to the first equation (of the three equations above)? It is not necessary to solve the full system.

4. Suppose $(X_t : t \in \mathbb{Z})$ is a stationary AR(1) process satisfying

$$X_t = \phi X_{t-1} + Z_t, \quad t \in \mathbb{Z},$$

where $(Z_t : t \in \mathbb{Z}) \sim$ IID $N(0, \sigma^2)$ and $|\phi| < 1$.

(a) What is the distribution of $X_t$ for fixed $t \in \mathbb{Z}$?

(b) What is the distribution of $X_{t+1}$ conditional on $X_t = x_t$?

(c) Suppose we observe a realization $(x_1, \ldots, x_n)$ of $(X_1, \ldots, X_n)$ for some $n \in \mathbb{N}$. What is the corresponding unconditional log likelihood function for $\phi$ and $\sigma^2$?

The conditional log likelihood function is, up to irrelevant additive constants, given by

$$l_c(\phi, \sigma^2) = -(n-1)\log\sigma - \sum_{i=2}^{n} \frac{(x_i - \phi x_{i-1})^2}{2\sigma^2}.$$

(d) Determine the joint conditional maximum likelihood estimator (MLE) $(\widehat{\phi}, \widehat{\sigma}^2)$ for $\phi$ and $\sigma^2$. (Hint: First determine the conditional MLE $\widehat{\sigma}^2$ for $\sigma^2$, expressed in terms of the conditional MLE $\widehat{\phi}$ for $\phi$.)

5. GARCH models are often used in the analysis of financial time series.

(a) List two qualitative properties of GARCH models which make these a suitable choice for modelling financial time series.

(b) What is the general formulation a GARCH($p$,$q$) model for the process $(X_t)$ with conditional variance process $\sigma_t^2$? Also specify, as general as possible, the relation between $X_t$ and $\sigma_t^2$.

Suppose $(X_t)$ is a stationary ARCH(1) process, with $X_t = \sigma_t Z_t$, where $(Z_t : t \in \mathbb{Z}) \sim$ WN$(0, 1)$, and where $(\sigma_t^2 : t \in \mathbb{Z})$ satisfies

$$\sigma_t^2 = \alpha + \phi X_{t-1}^2, \quad t \in \mathbb{Z}.$$

(c) Determine $\mathbb{E}X_t^2$.

(d) Show that there exists a white noise process $W_t$ such that $(X_t^2 : t \in \mathbb{Z})$ satisfies the equation

$$X_t^2 = \alpha + \phi X_{t-1}^2 + W_t,$$

and express $W_t$ in terms of $X_t$, $\sigma_t^2$ and/or $Z_t$.

*This was the final question.*