Delft University of Technology — Mekelweg 4
EWI Faculty — 2628 CD Delft

Time Series and Extreme Value theory WI4230
Final Exam - April 2017

Please provide your student card on the table, ready for inspection.
Mobiles, tablets and similar objects must be switched off. The exam is invalidated if you cheat.
Please write with a pen. Please write your name, surname and student number on all your papers.

The exam is on the extreme value theory part of the course. You have in total **3 hours for 4 questions.**

In the whole exam, the tail quantile function is denoted by $U(t)$, $t > 1$.

1. (6 points) Show that $F(x) = 1 - \exp(-\frac{x^2}{4})$, $x \geq 0$ is in a max domain of attraction. What is the extreme value index of this distribution?

2. Let F be a distribution function. Suppose that

$$\lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} = g(x)$$

exists and is continuous for $x > 0$; $0 < g(x) < \infty$ for any $x > 0$; and $g(2) = \frac{1}{2}$.

- (10 points) What is $g(x)$? Give the proof. Lemma $\lim_{t\to\infty} \frac{f(tx)}{f(t)} = x^\alpha$ niet gebruiken (of bewijzen)
- (4 points) Show that $\lim_{t\to\infty} \log \frac{U(tx)}{U(t)} = \log x$, for $x > 0$.
- (4 points) Let X be a random variable from F. Define $Y = \frac{1}{1-F(X)}$. What is the distribution function of Y? What is the distribution of $\log Y$?
- (10 points) Let $X_1, \ldots, X_n$ be a random sample from F. Let $k = k(n)$ be a sequence of integers such that $k \to \infty$ and $\frac{k}{n} \to 0$, as $n \to \infty$. Define

$$Z_n := \frac{1}{k}\sum_{i=1}^{k} \log X_{n-i+1,n} - \log X_{n-k,n}.$$

What is the limit distribution of Z_n, as $n \to \infty$? Give the proof.
Hint Let $\{E_i, 1 \leq i \leq n\}$ be a random sample from $EXP(1)$. Then

$$\{E_{n-i+1,n} - E_{n-j,n}, 1 \leq i \leq j\} \stackrel{d}{=} \{E^*_{j-i+1,j}, 1 \leq i \leq j\}$$

where $\{E^*_i, 1 \leq i \leq j\}$ is a random sample from $EXP(1)$, independent from $\{E_i, 1 \leq i \leq n\}$.

3. Let X be the daily maximum temperature in Delft in the summer season which lasts for 92 days from June 1st till August 31st. Let U be the tail quantile function of X: $\mathbf{P}(X > U(t)) = \frac{1}{t}$, for $t > 1$. Let $X_1, \ldots X_{920}$ be an identical and independent sample over 10 years.

- (4 points) Show that on average the temperature exceed $U(9200)$ once every 100 years. Here we assume that the temperature in the summer season is always higher than the temperature in other seasons. Thus it is sufficient to consider only the temperature in summer.
- (4 points) In order to estimate the 100-year return level, we assume that the distribution of X is in a max domain of attraction with extreme value index γ. Figure 1 displays the the scatter plot of the data. Figure 2 shows the estimates of the extreme value index by Hill and moment methods. Based on this, what can you conclude on the sign of γ?
- (6 points) Based on your conclusion on the sign of γ, derive the estimator of $U(9200)$. Note that the sufficient and necessary conditions for the max domain of attraction are given as below.

$$\lim_{t\to\infty} \frac{U(tx) - U(t)}{a(t)} = \frac{x^\gamma - 1}{\gamma}, \quad \text{for} \quad \gamma \in \mathbf{R}$$

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^\gamma, \quad \text{for} \quad \gamma > 0.$$

  If you choose to work with the first sufficient and necessary condition, it is allowed to denote the estimator of $a(\cdot)$ by $\hat{a}(\cdot)$ without giving the precise formula.
- (4 points) State two critical comments about the assumptions that we make on the data set in order to apply the estimation.

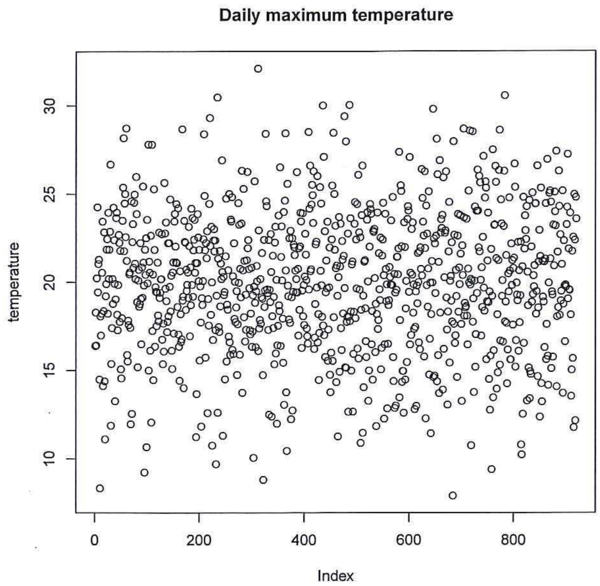

Figure 1: Scatter plot

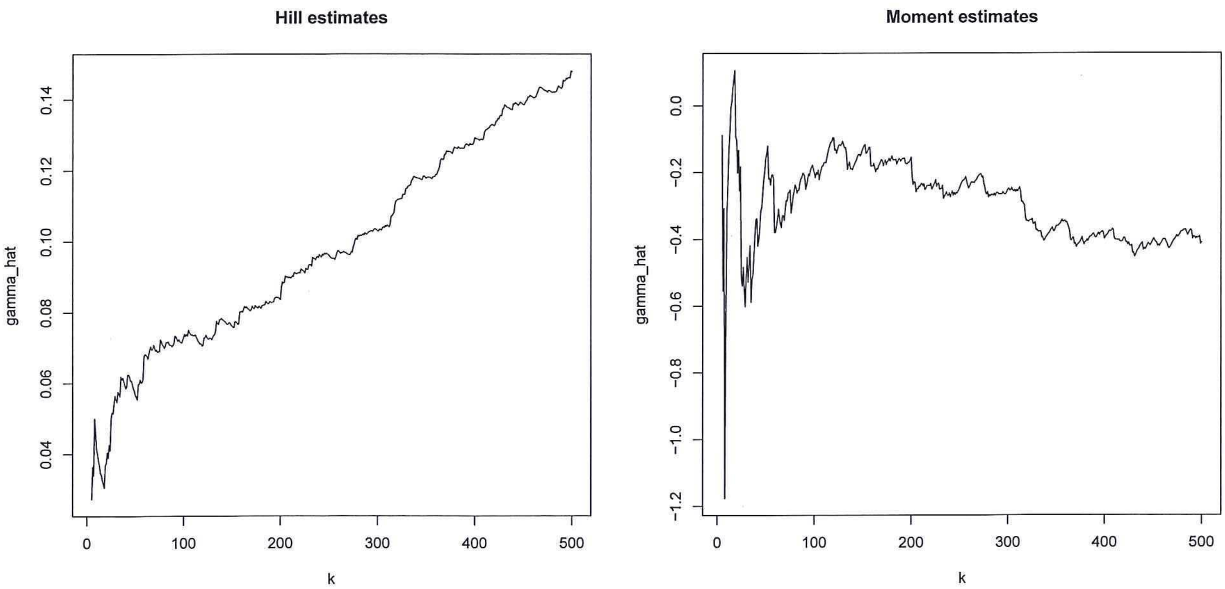

Figure 2: Hill estimates and moment estimates of γ

4. Let (X_1, X_2) be a bivariate random vector with a continuous distribution function F. Let U_1 and U_2 the tail quantile functions: $\Pr(X_i > U_i(t)) = \frac{1}{t}$, for $t \geq 1$ and $i = 1, 2$. Suppose there exists a non-degenerate distribution G_0 such that for $(x, y) \in (0, \infty] \times (0, \infty]$,

$$\lim_{n\to\infty} F^n(U_1(nx), U_2(ny)) = G_0(x, y).$$

Define $L(x, y) = -\log G_0(\frac{1}{x}, \frac{1}{y})$, for $x, y > 0$.

- (6 points) Show that $L(ax, ay) = aL(x, y)$, for any positive a, x and y.
- (4 points) Show that $\max(x, y) \leq L(x, y) \leq x + y$, for any positive x and y.
- (4 points) Give an example of (X_1, X_2) that corresponds to $L(x, y) = \max(x, y)$.
- (4 points) Give an example of (X_1, X_2) that corresponds to $L(x, y) = x + y$.