Delft University of Technology Mekelweg 4
EWI Faculty 2628 CD Delft

# Time Series and Extreme Value theory WI4230
# Final Exam - June 2016

Please provide your student card on the table, ready for inspection.
Mobiles, tablets and similar objects must be switched off. The exam is invalidated if you cheat.
Please write with a pen. Please write your name, surname and student number on all your papers.

The exam is made of two parts. You have in total 3 hours.
*If you sat for the Time Series' Partial in April, and you like your grade, you can skip the Time Series Part. In that case, you then have 1.5 hours*

***Good Luck!***

## Time Series Part

Please answer 4 out of the following 5 questions. They all have the same weight. You can choose the ones you prefer.

**1.** Consider the following points.

- Give the formal definitions of *strict stationarity* and *weak stationarity*.
- Provide at least one example of time series for each definition.
- List at least three possible causes of non-stationarity.

**2.** Consider the process

$$x_t = \beta_0 + \beta_1 t + w_t,$$

where $\beta_0$ and $\beta_1$ are fixed constants, and $w_t$ is a white noise with zero mean and variance $\sigma_w^2$.

- Is $x_t$ strictly and/or weakly stationary? Prove or disprove it.
- Prove that the first difference series $\nabla x_t = x_t - x_{t-1}$ is weakly stationary.
- Repeat the previous point, when $w_t$ is replaced by a general stationary process, say $y_t$, with mean function $\mu_y$ and autocovariance function $\gamma_y(h)$.

**3.** Define the process

$$x_t = \phi x_{t-1} + w_t,$$

where $|\phi| > 1$ and $w_t \sim WN(0, \sigma_w^2)$.

- Compute the autocovariance function of $x_t$.
- Verify that $x_t$ is not a causal process.
- Define a process $y_t$, equivalent to $x_t$, which is causal.

**4.** Using the operator representation, we define a process

$$(1 - 0.4B - 0.45B^2)x_t = (1 + B + 0.25B^2)w_t,$$

where $w_t$ is the usual white noise $w_t \sim WN(0, \sigma_w^2)$.

- What type of model is this? Is the above representation redundant? Motivate your answer.
- Is the process causal? Verify and, in case, give the corresponding representation.
- Is the process invertible? Verify and, in case, give the corresponding representation.

**5.** Let $w_t \sim WN(0, \sigma_w^2)$ be a white noise.

- Compute the spectrum (spectral density) of a TS $x_t$, where $x_t$ is a MA(1) of the form

$$x_t = w_t + \frac{1}{2} w_{t-1}.$$

- Compute the spectrum of a TS $x_t$, where $x_t$ is an AR(2) of the form

$$x_t = x_{t-1} - \frac{9}{10} x_{t-2} + w_t.$$

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**EVT Part**

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**1.** (6 points) Let $V$ be a uniform random variable on $[0, 1]$ and

$$X = \frac{\frac{1}{V}}{\log \frac{1}{V}}.$$

Show that $X \in D(G_1)$, that is $X$ being in a max domain of attraction with extreme value index 1.

**2.** Let $G_0(x) = \exp(-\exp(-x))$, $x \in R$.

- (5 points) Let $E_{1,n} \le E_{2,n} \le \ldots \le E_{n,n}$ be the order statistics of a random sample from $Exp(1)$. The distribution function of $Exp(1)$ is given by $F(x) = 1 - \exp(-x)$. Find $a_n > 0$ and $b_n \in R$ such that

$$\lim_{n\to\infty} \Pr\left(\frac{E_{n,n} - b_n}{a_n} \le x\right) = G_0(x).$$

- (5 points) Let $\{X_i, i = 1, \ldots, n\}$ be a random sample with distribution function $F(x) = 1 - x^{-\frac{1}{\gamma}}$, $x > 0$ and $\gamma > 0$. Prove that if $k = k(n) \to \infty$ and $k < n$, then as $n \to \infty$,

$$\frac{\log X_{n,n} - \log X_{n-k,n}}{\log k} \xrightarrow{P} \gamma$$

where $X_{1,n} \le X_{2,n} \le \ldots \le X_{n,n}$ are the order statistics.
**Hint:** You might use that $\{E_{n-i,k} - E_{n-k,n}\}_{i=0}^{k-1} =^d \{E^*_{k-i,k}\}_{i=0}^{k-1}$, where $\{E^*_1, \ldots, E^*_k\}$ is a random sample from $Exp(1)$.

**3.** Based on a random sample $X_1, \ldots, X_n$ from an unknown distribution $F$, one would like to build up an estimation procedure for the following probability,

$$\Pr(X_1 > x_0),$$

where $x_0$ is a given large value. Let $U$ be the tail quantile function: $\Pr(X_1 > U(t)) = \frac{1}{t}$, for $t \ge 1$. Assume the max domain of attraction condition on $F$, that is there exists a positive function $a(t)$ and $\gamma \in R$ such that for $x > 0$,

$$\lim_{t\to\infty} t(1 - F(a(t)x + U(t))) = (1 + \gamma x)^{-\frac{1}{\gamma}} \tag{1}$$

- (5 points) Show that (1) is equivalent to

$$\lim_{s\to x^*} \Pr\left(\frac{X_1 - s}{f(s)} > x | X_1 > s\right) = (1 + \gamma x)^{-\frac{1}{\gamma}}, \tag{2}$$

where $x^*$ is the right endpoint of $F$ and $f(t)$ a positive function.

- (4 points) Justify the following approximation

$$\Pr(X > x_0) \approx p\left(1 + \gamma \frac{x_0 - U(1/p)}{f(U(1/p))}\right)^{-\frac{1}{\gamma}}.$$

What conditions on $p$ are needed for this approximation?

- (5 points) For a chosen $p$, how do you estimate $\gamma$, $U(1/p)$ and $f(U(1/p))$? Describe the estimation procedure.

**4.** Let $(X_1, X_2)$ be a bivariate random vector with a continuous distribution function $F$. Let $U_1$ and $U_2$ the tail quantile functions: $\Pr(X_i > U_i(t)) = \frac{1}{t}$, for $t \geq 1$ and $i = 1, 2$. Suppose there exists a non-degenerate distribution $G$ such that for $(x, y) \in (0, \infty] \times (0, \infty]$,

$$\lim_{n\to\infty} F^n(U_1(nx), U_2(ny)) = G(x, y).$$

- (3 points) Compute $G(x, \infty)$, for $x > 0$.
- (3 points) Show that for any $(x, y)$ for which $0 < G(x, y) < 1$,

$$\lim_{n\to\infty} n\,(1 - F(U_1(nx), U_2(ny))) = -\log G(x, y).$$

- (4 points) Define $L(x, y) = -\log G(\frac{1}{x}, \frac{1}{y})$, $x, y > 0$.
  Show that for positive $x$ and $y$, $\max(x, y) \leq L(x, y) \leq x + y$.