Time Series and Extreme Value theory WI4230
Final Exam - June 2016

This is the solution on EVT part.

1. Let F^{-1} denote the quantile function of X. Then we have

$$X = F^{-1}(V).$$

Hence $F^{-1}(v) = \frac{\frac{1}{v}}{\log \frac{1}{v}}$.
None of you was able to see this. This is called probability integral transform. You might not know this name. But you should know that every continuous random variable can be linked to a uniform random variable via its quantile function. One learns this in the first year of probability course.

Define U the tail quantile function of X. Then $U(t) = F^{-1}(1 - \frac{1}{t}) = \frac{\frac{1}{1-\frac{1}{t}}}{\log \frac{1}{1-\frac{1}{t}}}$, for $t \geq 1$. For $x > 0$,

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = \lim_{t\to\infty} \frac{\frac{1}{1-\frac{1}{tx}}}{\log \frac{1}{1-\frac{1}{tx}}} \Big/ \frac{\frac{1}{1-\frac{1}{t}}}{\log \frac{1}{1-\frac{1}{t}}}$$
$$= \lim_{t\to\infty} \frac{\log \frac{1}{1-\frac{1}{t}}}{\log \frac{1}{1-\frac{1}{tx}}} = \lim_{t\to\infty} \frac{\log(1-\frac{1}{t})}{\log(1-\frac{1}{tx})} = x.$$

As long as you state the max domain condition (on F or on U) , you get two points.

2. During the exam, you were given $b_n = \log n$ and $a_n = 1$.

- $$\lim_{n\to\infty} \Pr\left(\frac{E_{n,n} - b_n}{a_n} \leq x\right) = \lim_{n\to\infty} F^n(a_n x + b_n) = (1 - e^{-(\log n + x)})^n$$
$$= \lim_{n\to\infty} (1 - ne^{-x})^n = e^{-e^{-x}}$$

  The last equality follows from that $\lim_{n\to\infty}(1 - x/n)^n = e^{-x}$. Some of you did it differently, which is also ok.

- This is a cleaned and simplified version of exercise 3.4 in the book, which is one of your homework. From the first question, we see that

$$E_{n,n} - \log n \xrightarrow{d} G_0, \text{ as } n \to \infty.$$

  Note that $\log X_i =^d \gamma E_i$ This you also saw in the proof on the consistency of Hill estimator.. Hence

$$\frac{\log X_{n,n} - \log X_{n-k,n}}{\log k} =^d \gamma \frac{E_{n,n} - E_{n-k,n}}{\log k} =^d \gamma \frac{E^*_{k,k}}{\log k}.$$

  From the first question we know that (since $k \to \infty$)

$$E^*_{k,k} - \log k \xrightarrow{d} G_0,$$

  which implies that

$$\frac{E^*_{k,k}}{\log k} \xrightarrow{p} 1.$$

  This concludes the proof.

3. This one we did in the class more than once.

- First note that $x^* = U(\infty)$. Now (1) can be written as

$$\lim_{t\to\infty} \frac{\Pr(X > a(t)x + U(t))}{\Pr(X > U(t))} = (1+\gamma x)^{-\frac{1}{\gamma}}.$$

For $x > 0$, this is equivalent to

$$\lim_{t\to\infty} \Pr\left(\frac{X_1 - U(t)}{a(t)} > x | X_1 > U(t)\right) = (1+\gamma x)^{-\frac{1}{\gamma}}.$$

Now write $U(t) = s$. Then $\lim_{t\to\infty} U(t) = x^*$ and

$$\lim_{s\to x^*} \Pr\left(\frac{X_1 - s}{a(U^{inv}(s))} > x | X_1 > s\right) = (1+\gamma x)^{-\frac{1}{\gamma}}.$$

The proof is finished by letting $f(s) = a(U^{inv}(s))$.

- The approximation is done by taking $s = U(1/p)$ in the conditional probability. Since we need $s \to x^*$, p has to tend to zero. Or in other words, since s has to be close to the endpoint, p has to be close to zero.
- I expect to see moment method (for $\gamma \in R$) or MLE method (for $\gamma > -1$), that we discussed in the lecture. You don't get punished if you forget to specify the boundary condition on MLE.

4.

- For positive x,

$$G(x,\infty) = \lim_{n\to\infty} F^n(U_1(nx), U_2(\infty)) = \lim_{n\to\infty} (P(X_1 \le U_1(nx)))^n = \lim_{n\to\infty} (1 - 1/nx)^n = e^{-\frac{1}{x}}. \quad (1)$$

- we did in the class
  For any (x, y) for which $0 < G(x, y) < 1$,

$$\lim_{n\to\infty} n \log F(U_1(nx), U_2(ny)) = \log G(x,y). \quad (2)$$

$$\lim_{n\to\infty} n \log(1 + F(U_1(nx), U_2(ny)) - 1) = \log G(x,y). \quad (3)$$

$$\lim_{n\to\infty} n(F(U_1(nx), U_2(ny)) - 1)) = \log G(x,y). \quad (4)$$

- When we discussed the properties of L function, i showed the proof for the homogeneity and asked you to prove the bounds yourself.
  By definition

$$L(x,y) = \lim_{n\to\infty} n(1 - F(U_1(n/x), U_2(n/y))) = \lim_{n\to\infty} n \Pr(X_1 > U_1(n/x) \text{ or } X_2 > U_2(n/y)).$$

Hence $L(x,y) \le \lim_{n\to\infty} n\left(\Pr(X_1 > U_1(n/x)) + \Pr(X_2 > U_2(n/y))\right) = x + y$.
Obviously, $L(x,y) \ge \lim_{n\to\infty} n \Pr(X_1 > U_1(n/x)) = x$.