Delft University of Technology
Faculty EEMCS
Mekelweg 4, 2628 CD Delft

Exam part 1 Real Analysis (TW2090)
6-11-2017; 13.30–15.30
Teacher M.C. Veraar, co-teacher K.P. Hart.

---

1. Let (M, d) and (N, ρ) be metric spaces and let $f : M \to N$ be a function.

(5) a. Complete the following definition: f is *continuous* in $x \in M$ if ....

(10) b. Assume that f is continuous in $x \in M$. Let (x_n) be a sequence in M with $x_n \xrightarrow{d} x$. Show that $f(x_n) \xrightarrow{\rho} f(x)$.

(5) 2. a. Let (M, d) be a metric space. Complete the following definition: a subset $A \subseteq M$ is called *totally bounded* if ...

(10) b. Give an example of a metric space (M, d) and a bounded subset $A \subseteq M$ which is not totally bounded. As always: prove all your assertions.

3. Let (M, d) be a metric space.

(5) a. Give the definition of the closure $\overline{A}$ of a set $A \subseteq M$.

(10) b. Using only the definition of the closure, prove the following equivalence for a set $A \subseteq M$: $\overline{A} = M \iff$ for all $x \in M$ and for all $\varepsilon > 0$ there exists a $y \in A$ such that $d(x, y) < \varepsilon$.

For a set $A \subseteq M$ and $\varepsilon > 0$ we define

$$A(\varepsilon) = \{x \in M : \exists y \in A \text{ such that } d(x, y) < \varepsilon\}.$$

(5) c. Show that $A(\varepsilon)$ is open.

Let (A_n) be a sequence of subsets of M such that for all $n \geq 1$ one has $A_n \subseteq A_{n+1}$ and $\overline{\bigcup_{n\geq 1} A_n} = M$.

(6) d. Use (b) to show that for each $\varepsilon > 0$ one has $M = \bigcup_{n\geq 1} A_n(\varepsilon)$.

(6) e. From now on assume that M is compact. Show that for every $\varepsilon > 0$ there exists an $n \geq 1$ such that $M = A_n(\varepsilon)$.

4. Let X be a nonempty set and let $B(X)$ be the vector space of bounded functions $f : X \to \mathbb{R}$. On $B(X)$ we define $\|f\|_\infty = \sup_{x\in X} |f(x)|$.

(6) a. Show that $\|\cdot\|_\infty$ is a norm on $B(X)$.

(12) b. Prove that $(B(X), \|\cdot\|_\infty)$ is complete.

(10) 5. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that for every integer $n \geq 0$,

$$\int_0^1 f(x)x^n dx = 0.$$

Show that $f = 0$.
Hint: First explain why for all polynomials p one has $\int_0^1 f(x)p(x)dx = 0$ and then use Weierstrass' theorem to find that $\int_0^1 (f(x))^2 dx = 0$.

The value of each (part of a) problem is printed in the margin; the final grade is calculated using the following formula

$$\text{Grade} = \frac{\text{Total} + 10}{10}$$

and rounded in the standard way.

THE END

1. a. for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $y \in M$: $d(x, y) < \delta$ implies $\rho(f(x), f(y)) < \varepsilon$.

   b. Let $\varepsilon > 0$. Since f is continuous we can find $\delta > 0$ such that for all $y \in M$: $d(x, y) < \delta$ implies $\rho(f(x), f(y)) < \varepsilon$. Since $x_n \to x$, we can find $N \in \mathbb{N}$ such that for all $n \geq N$, $d(x, x_n) < \delta$. It follows that $\rho(f(x), f(x_n)) < \varepsilon$.

2. a. ... for all $\varepsilon > 0$ there exist $x_1, \ldots, x_n \in M$ such that $A \subseteq \bigcup_{i=1}^n B_\varepsilon(x_i)$.

   b. Let $M = \mathbb{R}$ with the discrete metric. Then $B_2(0) = \mathbb{R}$ thus $\mathbb{R}$ is bounded. However, letting $\varepsilon = 1$, we find that for every $x \in \mathbb{R}$ we have $B_1(x) = \{x\}$. Thus for any choice $x_1, \ldots, x_n \in \mathbb{R}$, $\bigcup_{i=1}^n B_\varepsilon(x_i) = \{x_1, \ldots, x_n\}$ is not equal to $\mathbb{R}$. This shows that $\mathbb{R}$ with the discrete metric is bounded but not totally bounded.

3. a. This is the smallest closed set in M which contains A. In other words: $\overline{A} = \bigcap\{F \subseteq M : A \subseteq F \text{ and } F \text{ is closed}\}$.

   b. $\Leftarrow$ using contraposition. Assume $\overline{A} \neq M$. We will show that there exist $x \in M$ and $\varepsilon > 0$ such that for all $y \in A$, $d(x, y) \geq \varepsilon$. Choose $x \in M \setminus \overline{A}$. Since $\overline{A}$ is closed we have that $M \setminus \overline{A}$ is open. Thus we can $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq M \setminus \overline{A}$. Then we find $A \subseteq \overline{A} \subseteq M \setminus B_\varepsilon(x)$. Thus for every $y \in A$, $d(x, y) \geq \varepsilon$.
   $\Rightarrow$. Assume that for all $x \in M$ and for all $\varepsilon > 0$ there exists a $y \in A$ such that $d(x, y) < \varepsilon$. We will show that $\overline{A} = M$. For this choose $x \in M$ arbitrary and let $F \subseteq M$ be a closed set such that $A \subseteq F$. It suffices to show that $x \in F$. If $x \in M \setminus F$, then since $M \setminus F$ is open we can find an $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq M \setminus F$. Therefore, $B_\varepsilon(x) \cap A \subseteq B_\varepsilon(x) \cap F = \emptyset$. However, from the assumption we know that there exists a $y \in A$ such that $d(x, y) < \varepsilon$. Thus $y \in B_\varepsilon(x) \cap A$ and hence the latter is nonempty. This contradiction implies that we must have $x \in F$.

   c. Let $x \in A_\varepsilon$. We need to find a $\delta > 0$ such that $B_\delta(x) \subseteq A_\varepsilon$. Choose $y \in A$ such that $d(x, y) < \varepsilon$. Let $\delta = \varepsilon - d(x, y)$. Then $\delta > 0$ and for all $z \in B_\delta(x)$ we have by the triangle inequality,
   $$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \delta = \varepsilon.$$

   d. Fix $\varepsilon > 0$. Choose $x \in M$ arbitrary. Let $y \in \bigcup_{n\geq 1} A_n$ be such that $d(x, y) < \varepsilon$. Then we can find $n \in \mathbb{N}$ such that $y \in A_n$. Thus $y \in A_n(\varepsilon)$. Therefore, we can conclude $x \in \bigcup\limits_{n\geq 1} A_n(\varepsilon)$.

   e. By d and c we now that $(A_n(\varepsilon))_{n\geq 1}$ is an open cover of M. The compactness of M now implies that it has a finite subcover. Therefore, there exists a finite set $F \subseteq \mathbb{N}$ such that $M \subseteq \bigcup_{n\in F} A_n(\varepsilon)$. Since $A_n \subseteq A_{n+1}$ we also have $A_n(\varepsilon) \subseteq A_{n+1}(\varepsilon)$. Taking $N = \max F$, we find that $\bigcup_{n\in F} A_n(\varepsilon) = A_N(\varepsilon)$. We can now conclude that $M = A_N(\varepsilon)$.

4. a. Let $f \in B(X)$. Since f is bounded we know that for every $x \in X$, $0 \leq |f(x)| \leq M$. Therefore, $\|f\|_\infty = \sup_{x\in X} |f(x)|$ is a number in $[0, \infty)$. We check the remaining properties of a norm. If $f = 0$, then clearly, $\|f\|_\infty = 0$. Conversely, if $\|f\|_\infty = 0$, then $\sup_{x\in X} |f(x)| = 0$, thus $|f(x)| = 0$ for all $x \in X$, thus $f(x) = 0$ for all $x \in X$. If $\lambda \in \mathbb{R}$, then
   $$\|\lambda f\|_\infty = \sup_{x\in X} |\lambda f(x)| = \sup_{x\in X} |\lambda|\,|f(x)| = |\lambda| \sup_{x\in X} |f(x)| = |\lambda|\,\|f\|_\infty.$$
   where the numbers $|\lambda|$ can be pulled out of the supremum since it is in $[0, \infty)$. Finally, if $f, g \in B(X)$, then for all $x \in X$,
   $$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\|_\infty + \|g\|_\infty.$$
   Therefore, $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

   b. Finally, we prove the completeness of $B(X)$. Let $(f_n)_{n\geq 1}$ be a Cauchy sequence in $B(X)$.
   (i): We claim that for all $x \in X$, $(f_n(x))_{n\geq 1}$ is a Cauchy sequence in $\mathbb{R}$. Indeed, let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that for all $m, n \geq N$, $\|f_n - f_m\|_\infty < \varepsilon$. Then for all $x \in X$, for all $m, n \geq N$,
   $$|f_m(x) - f_m(x)| \leq \|f_n - f_m\|_\infty < \varepsilon \quad (*)$$
   which proves the claim. By the completeness of $\mathbb{R}$ we can define $f : X \to \mathbb{R}$ as $f(x) = \lim_{n\to\infty} f_n(x)$.
   (ii): Since $(f_n)_{n\geq 1}$ is a Cauchy sequence in $B(X)$ it is bounded in $B(X)$. Choose M such that for all $n \geq 1$, $\|f_n\|_\infty \leq M$. Then for all $x \in X$, $|f_n(x)| \leq M$. Therefore, letting $n \to \infty$, we find that for all $x \in X$, $|f(x)| \leq M$. Thus $f \in B(X)$.
   (iii): It remains to show that $f_n \to f$ in $B(X)$. Let $\varepsilon > 0$. Choose N as in step (i). Letting $m \to \infty$ in $(*)$ we obtain that for all $n \geq N$, for all $x \in X$,
   $$|f(x) - f_n(x)| \leq \varepsilon.$$
   Therefore, for all $n \geq N$, $\|f - f_n\|_\infty \leq \varepsilon$.

See also the next page.

5. Let $p(x) = \sum_{m=0}^{M} a_m x^m$ be a polynomial. Then by linearity of the integral we can write

$$\int_0^1 f(x)p(x)dx = \sum_{m=0}^{M} a_m \int_0^1 f(x)x^m dx = 0.$$

By Weierstras' theorem we can find polynomials $(p_j)_{j\geq 1}$ such that $p_j \to f$ uniformly on $[0, 1]$. Therefore, by the above observation and the standard properties of integrals, we have

$$\begin{aligned}\int_0^1 (f(x))^2 dx &= \Big| \int_0^1 (f(x))^2 dx - \int_0^1 f(x)p_j(x)dx \Big| \\ &= \Big| \int_0^1 (f(x))^2 - f(x)p_j(x)dx \Big| \\ &\leq \int_0^1 |(f(x))^2 - f(x)p_j(x)|dx \\ &= \int_0^1 |f(x)|\,|f(x) - p_j(x)|dx \\ &\leq \|f\|_\infty \|f - p_j\|_\infty.\end{aligned}$$

Since the right-hand side tends to zero as $j \to \infty$, we must have $\int_0^1 (f(x))^2 dx = 0$. Since f is continuous it follows that $f = 0$.

THE END