Introduction to Mathematical Finance (wi3417tu)
27 januari 2015, 18.30–21.30 uur

(No books, no notes.)

Please note: answers should be supplemented by motivation, explanation and/or calculation, whichever may be appropriate; you may choose Dutch or English as the language to use for your answers.
Point distribution: parts **1b** and **4a** each are worth 0.5 points; all other *parts* of a question are worth 1 point; the grade is the number of points earned plus 1.

1. Given is a binomial model with $S_0 = 8$, $u = 2$, $d = 3/4$, $p = q = 1/2$ and $r = 0$. You have 6 Euro at your disposal to trade with for two periods. Your *utility* function is $U(x) = -1/x$.

- **a.** What is your expected utility at $n = 2$ under optimal portfoliomanagement?
- **b.** What position should you take in the stoch at $n = 0$ to accomplish this?

2. Consider the standard binomial model for the stock price-evolution with $u = 2$, $d = 1/2$, $r = 1/4$ en $S_0 = 4$.

- **a.** Determine the value V_0 of an American put with expiration $n = 4$ and strike 4.
- **b.** What happens with V_0 if expiration is at $n = 5$ instead of $n = 4$: does it go up, down, or stay the same?
- **c.** An American chooser option allows you to choose, at time 1, to have either an American call or an American put with strike 4. Determine the value of an American chooser option expiring at time 4.

3. Consider the N-period binomial interest rate model, with interest rate process $R_0, \dots, R_{N-1}$ and probability measure $\tilde{\mathbb{P}}$.

- **a.** The *discount process* is denoted by $D_0, \dots, D_N$. State the definition of D_n and show that the definition
$$B_{n,m} = \tilde{\mathbb{E}}_n \left[\frac{D_m}{D_n} \right]$$
implies that $D_n B_{n,m}$, $n = 0, 1, \dots, m$ is a martingale.
- **b.** In the textbook it is proved that it follows from the definition of $B_{n,m}$ that every *discounted wealth process* is a martingale. This, in turn, implies that no arbitrage is possible; show this.
- **c.** Given an *asset price process* $S_0, \dots, S_N$ within this model. A theorem from the book says: if *zero-coupon bonds* of all maturities can be traded, then the m-forward price of this asset satisfies:
$$\mathrm{For}_{n,m} = \frac{S_n}{B_{n,m}}.$$
Provide a portfolio/arbitrage argument that demonstrates that this *has* to be the price.

4. Consider the 3-period interest rate model with risk-neutral probabilities, where the interest rates are path-dependent:

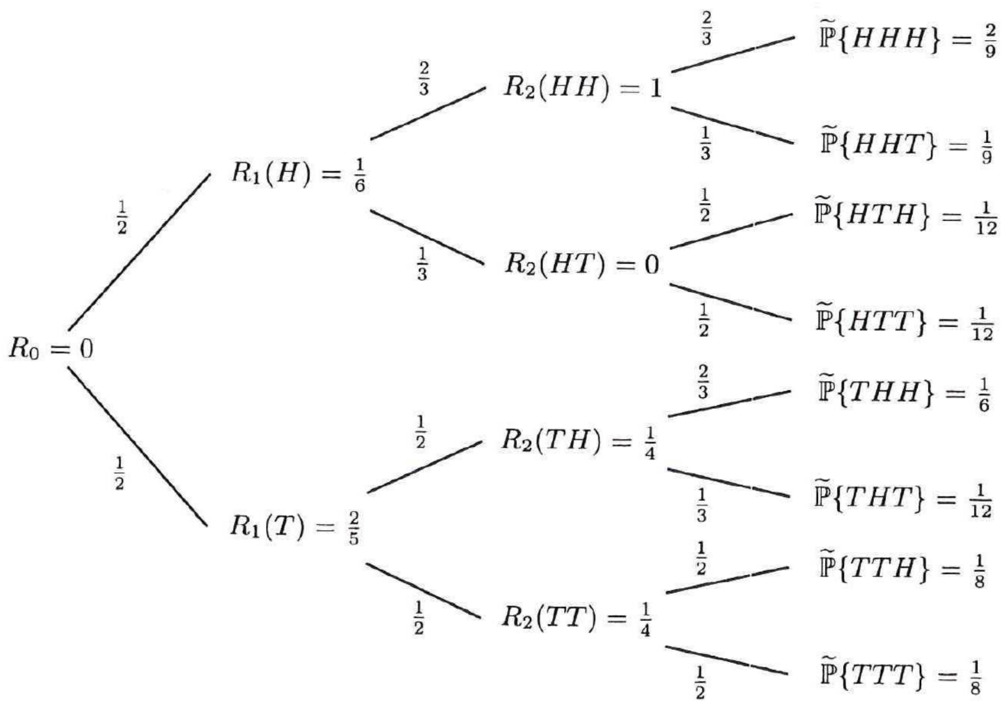

Fig. 6.3.1. A three-period interest rate model.

$\omega_1\omega_2$	$\frac{1}{1+R_0}$	$\frac{1}{1+R_1}$	$\frac{1}{1+R_2}$	D_1	D_2	D_3	$\widetilde{\mathbb{P}}$
HH	1	6/7	1/2	1	6/7	3/7	1/3
HT	1	6/7	1	1	6/7	6/7	1/6
TH	1	5/7	4/5	1	5/7	4/7	1/4
TT	1	5/7	4/5	1	5/7	4/7	1/4

You are being asked to determine some prices within this model.

a. Determine $B_{1,3}$.

b. Explain what a *3-period interest rate cap* is. Determine the price if $K = 1/4$ is the maximal interest to be paid.

Beknopte antwoorden:

1a -0.135

1b $X_1(H) = 10$, $X_1(T) = 5$, whence $\Delta_0 = 0.5$.

2a $V_0 = 0.928$.

2b New contract facilities arise, so the price cannot go down. In case $S_4 = 4$ it pays to continue, the continuation value is positive, while the intrinsic values is zero. The new $V_4(TTHH)$ increases wrt the old value, so V_0 increases.

2c Using the at a) computed values and the values for the call at time 1, which equal those of a European call, we find: $V_1(H) = 6.144$ (by choosing the call) and $V_1(T) = 2$ (put). So the chooser is worth 3.258.

3a See Shreve, pag 145 en 151.

3b See Shreve, Remark 6.2.7.

3c In essence, one is asked for the reasoning in the proof of Theorem 6.3.2, but also the reasoning in Remark 6.3.3 received full marks.

4a See Example 6.3.9.

4b See Example 6.3.9, but $K = \frac{1}{3}$, resulting in *time-zero* prices of the three caplets of 0, 3/56 and 3/28. The price of the interest rate cap therefore is $9/56 = 0.1607$.