TU Delft
Calculus TI1106M – Final test
Monday 30-01-2017, 9:00 – 11:00

NAME: STUDENT ID:

Remarks:

- Formula sheet is allowed, calculators are not.
- In every problem, results from earlier problems can be used.

Short answer problems
Fill in anwers on sheet

4pt 1. Let $z = \dfrac{2\mathrm{i}}{4-\mathrm{i}}$.
Rewrite z in the form $a + b\mathrm{i}$ with a and b real.

Solution:
Multiply numerator and denominator by $4 + \mathrm{i}$ and simplify:

$$\frac{2\mathrm{i}}{4-\mathrm{i}} = \frac{2\mathrm{i}\cdot(4+\mathrm{i})}{(4-\mathrm{i})(4+\mathrm{i})} = \frac{-2+8\mathrm{i}}{17} = \frac{-2}{17} + \frac{8}{17}\mathrm{i}.$$

2. Consider the following sequence: $a_n = \dfrac{3n-1}{\sqrt{2+n^2}}$ for $n \geq 1$ and integer.

2pt a. Indicate whether it is increasing, decreasing or neither.

Solution:
Note that $a_n = f(n)$ where $f : \mathbb{R} \to \mathbb{R}, x \mapsto \frac{3x-1}{\sqrt{2+x^2}}$. We have:

$$f'(x) = \frac{3\sqrt{2+x^2} - (3x-1)\frac{x}{\sqrt{2+x^2}}}{2+x^2} = \frac{6+x}{(2+x^2)^{\frac{3}{2}}}.$$

This is positive for $x > -6$. It follows that the sequence is increasing.

2pt b. Find, if possible, $\lim_{n\to\infty} a_n$.
In case of divergence, write DIV.

Solution:
The limit can be evaluated by division by the highest power (do not try l'Hospital!).:

$$\lim_{n\to\infty} \frac{3n-1}{\sqrt{2+n^2}} = \lim_{n\to\infty} \frac{3-\frac{1}{n}}{\sqrt{\frac{2}{n^2}+1}} = 3.$$

In the second step we used that $n = \sqrt{n^2}$, which is true since $n > 0$.

4pt 3. Given vectors $\mathbf{v} = \langle 2, -3, -1\rangle$ and $\mathbf{w} = \langle 2, h, 2\rangle$.
Find h such that $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.

Solution:
Vectors are orthogonal precisely if the dot product is 0. We find:

$$\mathbf{v}\cdot\mathbf{w} = 4 - 3h - 2 = 2 - 3h,$$

which vanishes for $h = \frac{2}{3}$. So $\mathbf{v}$ and $\mathbf{w}$ are orthogonal precisely for $h = \frac{2}{3}$.

4. Let function f be given by $f(x, y) = \sqrt{x^2 + xy}$, and point $P = (2, 6)$.

2pt a. Find $\nabla f(P)$.

Solution:
Find the partial derivatives:

$$f_x(x, y) = \frac{2x + y}{2\sqrt{x^2 + xy}}$$
$$f_y(x, y) = \frac{x}{2\sqrt{x^2 + xy}}.$$

Evaluate at P:

$$\nabla f(P) = \left\langle \frac{10}{8}, \frac{2}{8} \right\rangle = \left\langle \frac{5}{4}, \frac{1}{4} \right\rangle.$$

2pt b. Find the directional derivative of f at P in the direction $\mathbf{v} = \langle 1, -2 \rangle$.

Solution:
Let $\mathbf{u} = \frac{1}{|\mathbf{v}|}\mathbf{v} = \frac{1}{\sqrt{5}}\langle 1, -2 \rangle$.
The directional derivative is given by:

$$D_{\mathbf{u}} f(P) = \nabla f(P) \cdot \mathbf{u} = \frac{3}{4\sqrt{5}}.$$

2pt c. Use a linearization to approximate $f(2.04, 5.96)$

Solution:
The linearization at P is given by:

$$L(x, y) = f(P) + f_x(P)(x - 2) + f_y(P)(y - 6) = 4 + \frac{5}{4}(x - 2) + \frac{1}{4}(y - 6).$$

We have:

$$f(2.04, 5.98) \approx L(2.04, 5.98) = 4 + \frac{5}{4}(0.04) + \frac{1}{4}(-0.04) = 4.04.$$

Open problems

Provide calculations and argumentation!

1. Consider the power series $\sum_{n=1}^{\infty} \frac{1}{n^3\, 3^{n+1}}(2x-5)^n$.

6pt a. Show that the radius of convergence is $\frac{3}{2}$ and the center of convergence is $\frac{5}{2}$.

Solution:
We use the ratio test to investigate convergence:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{n^3}{(n+1)^3}\frac{1}{3}|2x-5| \to \frac{1}{3}|2x-5|$$

as $n \to \infty$.
It follows that the interval of convergence excluding the boundary points is given by all x such that $-1 < \frac{1}{3}(2x-5) < 1$, or equivalently: $1 < x < 4$.
The center of this interval is $(1+4)/2 = \frac{5}{2}$, the radius of convergence, i.e. the distance from center to boundary, is given by $(4-1)/2 = \frac{3}{2}$.

5pt b. Find the interval of convergence.

Solution:
We only have to investigate the boundary points.
At $x = 1$ we have the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n^3}$, at $x = 4$ we have the series $\sum_{n=1}^{\infty} \frac{1}{3n^3}$.
Both have $\sum_{n=1}^{\infty} \frac{1}{3n^3}$ as corresponding absolute series. This series is convergent, since it is a p-series with $p = 1 > 1$. It follows that the power series is absolutely convergent, and hence convergent, at both boundary points of the interval.
The interval of convergence is $[1, 4]$.

4pt 2. a. Write the complex number $-\frac{1}{3} - \frac{\mathrm{i}}{\sqrt{3}}$ in polar form.

Solution:
$-\frac{1}{3} - \frac{\mathrm{i}}{\sqrt{3}} = r(\cos(\theta) + \mathrm{i}\sin(\theta))$, where

$$r = \sqrt{\frac{1}{9} + \frac{1}{3}} = \sqrt{\frac{4}{9}} = \frac{2}{3},$$
$$\theta = \arctan(\sqrt{3}) + k\pi = \frac{1}{3}\pi + k\pi.$$

Since the complex number lies in the second quadrant, the argument is $\theta = \frac{4}{3}\pi$.

5pt b. Find all solutions to the equation $z^2 = -\frac{1}{3} - \frac{\mathrm{i}}{\sqrt{3}}$.

Provide your answers in the form $a + \mathrm{i}b$ with a and b in $\mathbb{R}$.

Solution:
Write $z = r(\cos(\theta) + \mathrm{i}\sin(\theta))$, then $z^2 = r^2(\cos(2\theta) + \mathrm{i}\sin(2\theta))$. The equation comes down to:

$$r^2 = \frac{2}{3}$$
$$2\theta = \frac{4}{3}\pi + 2k\pi.$$

It follows that $r = \sqrt{\frac{2}{3}}$ and $\theta = \frac{2}{3}\pi$ or $\theta = \frac{5}{3}\pi$. The solutions we get are:

$$z = \pm\sqrt{\frac{2}{3}}\left(-\frac{1}{2} + \frac{1}{2}\sqrt{3}\mathrm{i}\right) = \pm\left(-\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{2}}\mathrm{i}\right).$$

4pt 3. a. In class it was shown that $\cos(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$ for all $x \in \mathbb{R}$.

Use this to show that (for $x \geq 0$):

$$\int x^2 \cos(\sqrt{x})\,dx = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+3}}{(n+3)(2n)!} = C + \frac{1}{3}x^3 - \frac{1}{4\cdot 2!}x^4 + \frac{1}{5\cdot 4!}x^5 - \frac{1}{6\cdot 6!}x^6 + \ldots,$$

where C is an arbitrary constant.

Solution:

We have

$$\cos(y) = \sum_{n=0}^{\infty}(-1)^n \frac{y^{2n}}{(2n)!} = 1 - \frac{1}{2!}y^2 + \frac{1}{4!}y^4 - \frac{1}{6!}y^6 - \ldots \text{ for all } y \in \mathbb{R}.$$

Substitute $y = \sqrt{x}$ and multiply by x^2. This gives:

$$x^2 \cos(\sqrt{x}) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2+n}}{(2n)!} = x^2 - \frac{1}{2!}x^3 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \ldots \text{ for all } x \geq 0.$$

Integrate termwise:

$$\int x^2 \cos(\sqrt{x})\,dx = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{3+n}}{(3+n)(2n)!} = C + \frac{1}{3}x^3 - \frac{1}{4\cdot 2!}x^4 + \frac{1}{5\cdot 4!}x^5 - \frac{1}{6\cdot 6!}x^6 + \ldots.$$

6pt b. Use a. to approximate the integral $\int_0^1 x^2 \cos(\sqrt{x})\,dx$ with an error ≤ 0.01.

Motivate your answer!

Solution:

We can use the result from a. to evaluate the integral in terms of a series:

$$\begin{aligned}\int_0^1 x^2 \cos(\sqrt{x})\,dx &= \left[\sum_{n=0}^{\infty}(-1)^2 \frac{x^{3+n}}{(3+n)(2n)!}\right]_0^1 \\ &= \sum_{n=0}^{\infty}(-1)^2 \frac{1}{(3+n)(2n)!} \\ &= \frac{1}{3} - \frac{1}{2!\cdot 4} + \frac{1}{5\cdot 4!} - \frac{1}{6\cdot 6!} + \ldots\end{aligned}$$

The series is alternating, convergent, and $|a_{n+1}| < |a_n|$ for all terms. Therefore, for each partial sum s_k, we have that $|s - s_k| \leq |a_{k+1}|$, where s is the sum of the series. Note that the third term equals $\frac{1}{5\cdot 4!} = \frac{1}{120}$, so we can get an approximation with error ≤ 0.01 by using only the first 2 terms:

$$\int_0^1 x^2 \cos(\sqrt{x})\,dx \approx \frac{1}{3} - \frac{1}{8} = \frac{5}{24}.$$

4. Let f be a function of 2 variables given by $f(x, y) = y^3 - x^2 + \ln(y^3 x)$.
Let D be the maximal domain of f.

4pt a. Describe D and make a sketch. Clearly indicate or describe which points are part of D and which are not. Explain your answer.

Solution:

First quadrant and third quadrant, coordinate axes are *not* part of the domain.

4pt b. Show that $P = \left(-\sqrt{\frac{1}{2}}, -1\right)$ is a critical point of f.

Solution:

Find the partial derivatives of f:

$$f_x(x,y) = -2x + \frac{1}{x}$$
$$f_y(x,y) = 3y^2 + \frac{3}{y}$$

It is easy to check that both vanish at P. Therefore, P is a critical point of f.

6pt c. Explain whether f has a local maximum, local minimum or neither at P.

Solution:

To determine the type we need to find the Hessian (in Stewart section 14.7 denoted by D). For the Hessian, we need the second order partial derivatives:

$$f_{xx}(x,y) = -2 - \frac{1}{x^2}$$
$$f_{yy}(x,y) = 6y - \frac{3}{y^2}$$
$$f_{xy}(x,y) = 0$$

The Hessian at P is:

$$H(P) = f_{xx}(P)\, f_{yy}(P) \; - \; (f_{xy}(P))^2 = -4 \cdot -9 - 0^2 = 36.$$

This is larger than 0. Furthermore, $f_{xx}(P) < 0$. It follows that f attains a local maximum at P.

5pt d. Does f have any other critical points? Explain!

Solution:

We need to solve the system:

$$\begin{cases} f_x(x,y) = -2x + \frac{1}{x} = 0 \\ f_y(x,y) = 3y^2 + \frac{3}{y} = 0 \end{cases}$$

The first equation gives $x = \pm\sqrt{\frac{1}{2}}$, the second gives $y = -1$. We find two candidate points: point P and the point $(\sqrt{\frac{1}{2}}, -1)$. However, the latter point does not lie in the maximal domain of f. Therefore, f has P as its only critical point.

5. Consider the following iterated integral: $\displaystyle\int_{y=1}^{2} \int_{x=0}^{\sqrt{y}} f(x,y)\, dx\, dy.$

   Let D be the domain of integration.

2pt a. Sketch D.

6pt b. To reverse the order of integration, the integral has to be split:

$$\iint_D f(x,y) dA = \int_{\cdots}^{\cdots} \int_{\cdots}^{\cdots} f(x,y)\, dy\, dx + \int_{\cdots}^{\cdots} \int_{\cdots}^{\cdots} f(x,y)\, dy\, dx.$$

Give the limits for both integrals.

Solution:

$$\iint_D f(x,y) dA = \int_0^1 \int_1^2 f(x,y)\, dy\, dx + \int_1^{\sqrt{2}} \int_{x^2}^{2} f(x,y)\, dy\, dx.$$

6pt c. Evaluate $\iint_D y^{3/2}\cos(x\sqrt{y})dA$.

Use the order of integration that you think is most suitable.

Solution:

It is more convenient to integrate w.r.t. x first:

$$\begin{aligned}\int_{y=1}^{2}\int_{x=0}^{\sqrt{y}} y^{3/2}\cos(x\sqrt{y})\,dx\,dy &= \int_{y=1}^{2}[y\sin(x\sqrt{y})]_0^{\sqrt{y}}\,dy\\ &= \int_1^2 y\sin(y)\,dy\\ &= [-y\cos(y)]_1^2 + \int_1^2 \cos(y)\,dy\\ &= -2\cos(2)+\cos(1)+\sin(2)-\sin(1).\end{aligned}$$

$$\text{Grade} = 1 + \frac{\text{score}}{9}$$