Exam Reasoning and Logic (TI1306), 9:00 – 10:30

- Hint: First consider what would be the correct answer for a multiple choice question and then find that answer in the options given.

Multiple choice questions (1 point for each question)

1. If P and Q are properties of pairs of integers, in what way can you prove this theorem?

   Theorem. *For all $x \in \mathbb{Z}$ there exists a $y \in \mathbb{Z}$ so that $P(x, y) \to Q(x, y)$.*

   A. Take a random $x \in \mathbb{Z}$ and find a $y \in \mathbb{Z}$ (potentially dependent on x) for which it holds that from the assumption that $P(x, y)$ does not hold, it follows that $Q(x, y)$ does not hold either.

   B. Find a $y \in \mathbb{Z}$ and show for a random $x \in \mathbb{Z}$ that $P(x, y)$ follows from the assumption of $Q(x, y)$.

   C. Take a random x and y in $\mathbb{Z}$ and show that $Q(x, y)$ follows from the assumption of $P(x, y)$.

   D. Take a random $x \in \mathbb{Z}$ and find a $y \in \mathbb{Z}$ (potentially dependent on x) for which you show that from the assumption that $Q(x, y)$ does not hold, it follows that $P(x, y)$ does not hold.

   Answer: The theorem consists of a "for all" predicate followed by a "there exists" predicate. For the "for all" predicate we are required to take a random element of $\mathbb{Z}$ and prove that the theorem holds. For the "there exists" we can simply show that there is one, even a specific one will do. Finally we can choose to either prove the theorem as it is stated here, or prove the logically equivalent contrapositive ($\neg Q(x, y) \to \neg P(x, y)$). Answer D does exactly this.

2. If you want to prove the following theorem by mathematical induction, what do you have to do during the inductive step?

   Theorem. *For all integers $n \geq 1$ it holds that: $6 \mid (n^3 - n)$.*

   A. Find an integer k for which an integer a exists such that $(k^3 - k) = a \cdot 6$ holds, and show that for a random integer b it holds that $((k + 1)^3 - (k + 1)) = b \cdot 6$.

   B. Show for a random integer k that if there exists a random integer a so that $((k+1)^3 - (k+1)) = a \cdot 6$ holds, there exists an integer b so that $(k^3 - k) = b \cdot 6$ holds.

   C. Show that if it holds that when $n = 1$, there exists an integer a so that $(n^3 - n) = a \cdot 6$, then there also exists an integer b so that $((n + 1)^3 - (n + 1)) = b \cdot 6$.

   D. Show that if there exists an integer a such that $(k^3 - k) = a \cdot 6$ holds for a random integer k, there also exists an integer b such that $((k + 1)^3 - (k + 1)) = b \cdot 6$.

   Answer: The inductive step always requires you to prove that $P(k) \to P(k + 1)$ for a random k, or in English: if the integer k has the property, then so does the integer $k + 1$. If we take $P(k)$ as $6 \mid (k^3 - k)$, then we can reformulate the answers as such:

   A. Prove that: $\exists k(P(k) \to P(k + 1))$

   B. Prove that: $\forall k(P(k + 1) \to P(k))$

   C. Prove that: $P(1) \to P(2)$.

   D. Prove that: $\forall k(P(k) \to P(k + 1))$

   It is now obvious that D is exactly what we want.

3. If A, B and C are sets in the universe U, and $D = ((A \cup B)^c \cap (C - A))$, which of the following options is **not** true?
   A. $(A \cap B) \subseteq (C - D)$**.**
   B. $D \subseteq B^c$.
   C. $(A \cap B) \cap D = \emptyset$.
   D. $C^c \subseteq D^c$.

Answer: You can easily represent the set D as the set with all elements which are in not in $A \cup B$ but are in $C - A$. Make a Venn diagram with the three sets, and look at the part of C which isn't a part of A or B. Set D is everything *inside* that set.

A. This statement does not hold: There can be elements in C that are not part of both A *and* B, but are part of only one. This way they are not in D and thus they are in $C - D$, but they are not in $A \cap B$.

B. This statement holds. Drawing the Venn diagram will help you see that! Everything that is in D is a part of C but not a part of B (after all it is not in the union of A and B).

C. This statement holds. Everything that is in (A and B) is also in (A or B), thus not in D. Thus we find nothing that is in these three sets at the same time.

D. This statement holds. The contrapositive of this is that everything that is in D is also in C, which is easily seen from the definition of D.

4. If X, Y, Z and W are sets, how can you prove the following statement "If $X \subseteq Y$, then $Z \subseteq W$" ?
   A. By showing that $Z \subseteq X$ and that $Y \subseteq W$.
   B. By showing that $Y \subseteq Z$ and that $W \subseteq X$.
   C. By showing that $X \subseteq Z$ and that $Y \subseteq W$.
   D. By showing that $Z \subseteq X$ and that $W \subseteq Y$.

Answer: If $X \subseteq Y$ is true, then you get, if you can show that $Z \subseteq X$ and $Y \subseteq W$, the following relations: $Z \subseteq X \subseteq Y \subseteq W$. From this it follows that $Z \subseteq W$ because of transitivity.

5. Let A and B be sets for which it holds that $A \subseteq B$. Which of the following can we **not** say with certainty?
   A. If $B = \emptyset$ then $A = \emptyset$.
   B. $|A| < |B|$**.**
   C. If $A \neq \emptyset$ then $B \neq \emptyset$.
   D. $(A - B) = \emptyset$.

Answer:

A If there does not exist an element in B, there also does not exist an element in A, otherwise $A \not\subseteq B$.

B If $A = B$, $A \subseteq B$ holds, but not $|A| < |B|$.

C If there exists an element in A, that element also has to exist in B, otherwise $A \not\subseteq B$.

D If $A - B$ has an element then that means that A has an element B does not, thus $A \not\subseteq B$.

6. The following argument does not hold. Which sets can be used for a counterexample?

Argument. *For all sets A and B it holds that:* $2^A \cup 2^B = 2^{A \cup B}$.

A. $A = \emptyset$ and $B = \{a, b\}$.
B. $A = \{a, b\}$ and $B = \{a\}$.
C. $A = \{a\}$ and $B = \{b\}$.
D. $A = \{\emptyset\}$ and $B = \emptyset$.

Answer: This claim is true if $A \subseteq B$ or $B \subseteq A$, only for one of the pairs this is false: $A = \{a\}$ en $B = \{b\}$.

7. Consider the directed graph which corresponds to the (binary) relation R on a set A. If it is given that relation R **is** reflexive and **is not** symmetric, which of the following must hold?

A. (i) If there is an edge from a vertex to another vertex, there is no edge the other way, and (ii) there is an edge from every vertex to itself.
B. (i) Not all vertices have an edge to themselves, and (ii) if there is an edge from a vertex to another vertex, there is also an edge the other way.
C. (i) There are no edges between pairs of vertices, but (ii) every vertex has an edge to itself.
D. (i) There exists an edge from a vertex to another vertex, without an edge going the other way, and (ii) every vertex has an edge to itself.

Answer: Because R is reflexive, every vertex in G_R has an edge to itself. Because R is not symmetric, there has to be at least one edge for which the graph does not contain its counterpart.

8. Consider the following relation T on $\mathbb{Z}$: A pair $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ is an element of T if and only if $x \mid 2y$. Which of the following is true?

A. The relation T is reflexive and is symmetric.
B. The relation T is reflexive and is not symmetric.
C. The relation T is not reflexive and is symmetric.
D. The relation T is not reflexive and is not symmetric.

Answer: So T contains for example $(0, 0)$, $(-2, 1)$, $(-1, 1)$, $(1, 1)$, $(2, 1)$, $(-4, 2)$, $(-2, 2)$, $(-1, 2)$, $(1, 2)$, $(2, 2)$, $(4, 2)$, ... What are the properties of this relation?

Reflexive: The relation **is reflexive**, because for every integer x it holds that $x \mid 2x$.

Symmetric: The relation is **not symmetric**, because for every odd number $n > 1$, 2 divides $2n$, but an odd number greater than 1 does not divide $2 \cdot 2$. For example: $2 \mid 2 \cdot 5$, so $(2, 5) \in T$, but $\neg(5 \mid 2 \cdot 2)$, so $(5, 2) \notin T$.

Transitive: If we take random integers x, y and z, but in such a way that $(x, y) \in T$ and $(y, z) \in T$, then does $(x, z) \in T$ also always hold? The fact that $(x, y), (y, z) \in T$ means that there exist integers a and b such that $2y = ax$ and $2z = by$. Does this guarantee that there exists an integer c such that $2z = cx$? We know that $2z = by$, but $y = \frac{a}{2}x$, so $2z = b\frac{a}{2}x$, so $(x, z) \notin T$ if $b\frac{a}{2}$ is not an integer, which is the case if a and b are odd. Take for example $x = 4$, $y = 6$ en $z = 3$. Now $4 \mid 2 \cdot 6$ and $6 \mid 2 \cdot 3$ do hold, but it does not hold that $4 \mid 2 \cdot 3$, so the relation T is **not transitive**.

9. The functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are given as $f(x) = 3x^3 + 1$ and $g(x) = 2x^2 + 6x - 5$. Which of the following is true?

   A. The function f is one-to-one and the function g is one-to-one.

   B. The function f is one-to-one and the function g is not one-to-one.

   C. The function f is not one-to-one and the function g is one-to-one.

   D. The function f is not one-to-one and the function g is not one-to-one.

Answer: f is one-to-one as for every $f(a) = f(b)$ we find exactly one $x = a = b$ to solve it ($x = (\frac{1}{3}(f(a) - 1))^{\frac{1}{3}}$), but $g(x)$ is not: take $g(x) = -5$, this leads to $x = 0$ but also $x = -3$.

10. Consider the functions f and g again from the previous question (9). Which of these statements is true?

    A. The function f is onto and the function g is onto.

    B. The function f is onto and the function g is not onto.

    C. The function f is not onto and the function g is onto.

    D. The function f is not onto and the function g is not onto.

Answer: The previous answer already answers this question as well. But for every $y = f(x)$ we can find an x (using the formula given in the answer of question 9), whereas for $y = g(x)$ it is possible no x exists to make a y. For instance there is no x such that $g(x) = -100$.

Open questions

1. Consider the following arguments, where 2^X is the power set of the set X. (To make things clear: Epp writes 2^X as $\mathscr{P}(X)$ in her book.)

Argument (I). *For all sets A and B it holds that:* $2^{A-B} \not\subseteq (2^A - 2^B)$.

Argument (II). *For all sets A and B it holds that:* $(2^A - 2^B) \not\subseteq 2^{A-B}$.

Argument **I does hold**, and argument **II does not hold**.

(a) (1 point) Give for sets $C = \{0,1\}$ and $D = \{1\}$ the sets 2^{C-D} and $(2^C - 2^D)$ as an enumeration of elements (so in 'set-roster notation').

Answer:

$$C - D = \{0,1\} - \{1\} = \{0\}$$
$$2^{C-D} = \{\emptyset, \{0\}\} \qquad \text{first answer}$$
$$2^C = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$$
$$2^D = \{\emptyset, \{1\}\}$$
$$2^C - 2^D = \{\{0\}, \{0,1\}\} \qquad \text{second answer}$$

(b) (2 points) Give a proof for argument I. Explicitly note the proof techniques you use and explain all your steps. (You get the points for this question if your proof shows your *knowledge* of this matter.)

Answer:

Proof. You have to prove that for every set A and B it holds that: $2^{A-B} \not\subseteq (2^A - 2^B)$. So take random sets A and B. Now you have to prove that $2^{A-B} \not\subseteq (2^A - 2^B)$ is true, so we have to show that there exists an element in 2^{A-B} which does not exist in $(2^A - 2^B)$. Consider the empty set $\emptyset$. Because A and B are sets, $A - B$ is also a set, and the empty set is a subset of all sets, so $\emptyset \subseteq A - B$, so $\emptyset \in 2^{A-B}$. Because A and B are sets, and the empty set is a subset of all sets, $\emptyset \subseteq A$ and $\emptyset \subseteq B$ hold, so $\emptyset \in 2^A$ and $\emptyset \in 2^B$. In th set $2^A - 2^B$ are all elements of 2^A which are not in 2^B, so $\emptyset$ is not part of that, because that is an element of 2^B, so $\emptyset \notin (2^A - 2^B)$. This shows that $2^{A-B} \not\subseteq (2^A - 2^B)$. Because A and B are random sets, this holds for all sets A and B, which had to be proven. QED

(c) i. (1 point) Give a counterexample which shows that argument II does not hold.

Answer: The counterexample has to show that the statement "For all sets A and B it holds that: $(2^A - 2^B) \not\subseteq 2^{A-B}$" does not hold. Such a counterexample consists of specific sets A and B for which $(2^A - 2^B) \not\subseteq 2^{A-B}$ is **false**, because this shows that this statement does not hold *for all* sets A and B. So we have to find **two sets** A **and** B **for which** $(2^A - 2^B) \subseteq 2^{A-B}$ **is true**. A simple solution is to choose two sets for which $(2^A - 2^B) = \emptyset$, which we can accomplish for example by choosing $A = B$, or in general, to choose $A \subseteq B$. For this $2^A = 2^B$ (respectively $2^A \subseteq 2^B$), so there is no element in 2^A which is not in 2^B. So take as counterexample $A = B = \{1\}$.

ii. (1 point) Explain clearly and precisely how you prove the invalidity of the argument using your counterexample.

Answer: With the sets $A = B = \{1\}$, $2^A = 2^B = \{\emptyset, \{1\}\}$, so $(2^A - 2^B) = \emptyset$. Because the empty set is a subset of all sets, $\emptyset \subseteq 2^{A-B}$, which means that for the sets $A = \{1\}$ and $B = \{1\}$ it holds that $(2^A - 2^B) \subseteq 2^{A-B}$, which again shows that $(2^A - 2^B) \not\subseteq 2^{A-B}$ does not hold for all sets.

2. (5 points) Prove using mathematical induction that for all integers $n \geq 1$ it holds that: $\sum_{i=1}^{n}(i \cdot i\,!) = (n+1)\,! - 1$.

   Hint: Set up your proof such that it shows your *insight* in the steps of a proof by induction: The proof which you deliver should show indubitably that you understand how and why such a proof *works*! Give ample explanations and comments.

Answer: We use the predicate $P(n)$ to show that an integer n has the property $\sum_{i=1}^{n}(i \cdot i!) = (n+1)! - 1$. We have to prove that every integer $n \geq 1$ has this property.

Proof using mathematical induction. **Basis step ($n = 1$):** We need to prove that $P(1)$ is true, so that $\sum_{i=1}^{n}(i \cdot i!) = (n+1)! - 1$ is true if $n = 1$.

$$\sum_{i=1}^{1}(i \cdot i!) = (1 \cdot 1!) = 1 \cdot 1 = 1 = 2 - 1 = 2! - 1 = (1+1)! - 1,$$

so $P(1)$ is true.

Inductive step: In this part we need to prove that for all integers $n \geq 1$ it holds that: $P(n) \rightarrow P(n+1)$. So we take a random number ≥ 1, say k. We need to prove that the implication $P(k) \rightarrow P(k+1)$ is true. Suppose that $P(k)$ holds (this is the inductive hypothesis IH), so that number k has the property, so that $\sum_{i=1}^{k}(i \cdot i!) = (k+1)! - 1$. We need to prove that $P(k+1)$ holds, so that $\sum_{i=1}^{k+1}(i \cdot i!) = (k+1+1)! - 1$.

$$\begin{aligned}
\sum_{i=1}^{k+1}(i \cdot i!) &= \sum_{i=1}^{k}(i \cdot i!) + (k+1) \cdot (k+1)! && \text{split off last term} \\
&= (k+1)! - 1 + (k+1) \cdot (k+1)! && \text{according to the IH} \\
&= (k+1)! - 1 + (k+1)! \cdot (k+1) && ab = ba \\
&= (k+1)! \cdot (k+1) + (k+1)! - 1 && \text{reorder} \\
&= (k+1)! \cdot ((k+1) + 1) - 1 && \text{split off } (k+1)! \\
&= (k+2)! - 1 && (k+1)! \cdot (k+2) = (k+2)! \\
&= (k+1+1)! - 1,
\end{aligned}$$

So the number $k+1$ does indeed have the property: $P(k+1)$ is derived from the assumption that $P(k)$ is true, so the implication is proven for k. Because k is a random integer ≥ 1, the implication holds for all integers ≥ 1, which was proven in the inductive step.

According the principle of induction, we can conclude that $P(n)$ holds for all integers $n \geq 1$. QED

End of exam questions