Exam Reasoning and Logic (TI1306), 9:00 – 10:30

- Hint: First consider what would be the correct answer for a multiple choice question and then find that answer in the options given.

Multiple choice questions (1 point for each question)

1. If P and Q are properties of pairs of integers, in what way can you prove this theorem?

   Theorem. *For all $x \in \mathbb{Z}$ there exists a $y \in \mathbb{Z}$ so that $P(x, y) \to Q(x, y)$.*

   A. Take a random $x \in \mathbb{Z}$ and find a $y \in \mathbb{Z}$ (potentially dependent on x) for which it holds that from the assumption that $P(x, y)$ does not hold, it follows that $Q(x, y)$ does not hold either.

   B. Find a $y \in \mathbb{Z}$ and show for a random $x \in \mathbb{Z}$ that $P(x, y)$ follows from the assumption of $Q(x, y)$.

   C. Take a random x and y in $\mathbb{Z}$ and show that $Q(x, y)$ follows from the assumption of $P(x, y)$.

   D. Take a random $x \in \mathbb{Z}$ and find a $y \in \mathbb{Z}$ (potentially dependent on x) for which you show that from the assumption that $Q(x, y)$ does not hold, it follows that $P(x, y)$ does not hold.

2. If you want to prove the following theorem by mathematical induction, what do you have to do during the inductive step?

   Theorem. *For all integers $n \geq 1$ it holds that: $6 \mid (n^3 - n)$.*

   A. Find an integer k for which an integer a exists such that $(k^3 - k) = a \cdot 6$ holds, and show that for a random integer b it holds that $((k + 1)^3 - (k + 1)) = b \cdot 6$.

   B. Show for a random integer k that if there exists a random integer a so that $((k+1)^3 - (k+1)) = a \cdot 6$ holds, there exists an integer b so that $(k^3 - k) = b \cdot 6$ holds.

   C. Show that if it holds that when $n = 1$, there exists an integer a so that $(n^3 - n) = a \cdot 6$, then there also exists an integer b so that $((n + 1)^3 - (n + 1)) = b \cdot 6$.

   D. Show that if there exists an integer a such that $(k^3 - k) = a \cdot 6$ holds for a random integer k, there also exists an integer b such that $((k + 1)^3 - (k + 1)) = b \cdot 6$.

3. If A, B and C are sets in the universe U, and $D = ((A \cup B)^c \cap (C - A))$, which of the following options is **not** true?

   A. $(A \cap B) \subseteq (C - D)$.

   B. $D \subseteq B^c$.

   C. $(A \cap B) \cap D = \emptyset$.

   D. $C^c \subseteq D^c$.

4. If X, Y, Z and W are sets, how can you prove the following statement "If $X \subseteq Y$, then $Z \subseteq W$" ?

   A. By showing that $Z \subseteq X$ and that $Y \subseteq W$.

   B. By showing that $Y \subseteq Z$ and that $W \subseteq X$.

   C. By showing that $X \subseteq Z$ and that $Y \subseteq W$.

   D. By showing that $Z \subseteq X$ and that $W \subseteq Y$.

5. Let A and B be sets for which it holds that $A \subseteq B$. Which of the following can we **not** say with certainty?

   A. If $B = \emptyset$ then $A = \emptyset$.

   B. $|A| < |B|$.

   C. If $A \neq \emptyset$ then $B \neq \emptyset$.

   D. $(A - B) = \emptyset$.

6. The following argument does not hold. Which sets can be used for a counterexample?

   Argument. *For all sets A and B it holds that:* $2^A \cup 2^B = 2^{A \cup B}$.

   A. $A = \emptyset$ and $B = \{a, b\}$.
   B. $A = \{a, b\}$ and $B = \{a\}$.
   C. $A = \{a\}$ and $B = \{b\}$.
   D. $A = \{\emptyset\}$ and $B = \emptyset$.

7. Consider the directed graph which corresponds to the (binary) relation R on a set A. If it is given that relation R **is** reflexive and **is not** symmetric, which of the following must hold?

   A. (i) If there is an edge from a vertex to another vertex, there is no edge the other way, and (ii) there is an edge from every vertex to itself.
   B. (i) Not all vertices have an edge to themselves, and (ii) if there is an edge from a vertex to another vertex, there is also an edge the other way.
   C. (i) There are no edges between pairs of vertices, but (ii) every vertex has an edge to itself.
   D. (i) There exists an edge from a vertex to another vertex, without an edge going the other way, and (ii) every vertex has an edge to itself.

8. Consider the following relation T on $\mathbb{Z}$: A pair $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ is an element of T if and only if $x \mid 2y$. Which of the following is true?

   A. The relation T is reflexive and is symmetric.
   B. The relation T is reflexive and is not symmetric.
   C. The relation T is not reflexive and is symmetric.
   D. The relation T is not reflexive and is not symmetric.

9. The functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are given as $f(x) = 3x^3 + 1$ and $g(x) = 2x^2 + 6x - 5$. Which of the following is true?

   A. The function f is one-to-one and the function g is one-to-one.
   B. The function f is one-to-one and the function g is not one-to-one.
   C. The function f is not one-to-one and the function g is one-to-one.
   D. The function f is not one-to-one and the function g is not one-to-one.

10. Consider the functions f and g again from the previous question (9). Which of these statements is true?

    A. The function f is onto and the function g is onto.
    B. The function f is onto and the function g is not onto.
    C. The function f is not onto and the function g is onto.
    D. The function f is not onto and the function g is not onto.

Open questions

1. Consider the following arguments, where 2^X is the power set of the set X. (To make things clear: Epp writes 2^X as $\mathscr{P}(X)$ in her book.)

   Argument (I). *For all sets A and B it holds that:* $2^{A-B} \not\subseteq (2^A - 2^B)$.

   Argument (II). *For all sets A and B it holds that:* $(2^A - 2^B) \not\subseteq 2^{A-B}$.

   Argument **I does hold**, and argument **II does not hold**.

   (a) (1 point) Give for sets $C = \{0, 1\}$ and $D = \{1\}$ the sets 2^{C-D} and $(2^C - 2^D)$ as an enumeration of elements (so in 'set-roster notation').

   (b) (2 points) Give a proof for argument I. Explicitly note the proof techniques you use and explain all your steps. (You get the points for this question if your proof shows your *knowledge* of this matter.)

   (c) i. (1 point) Give a counterexample which shows that argument II does not hold.

   ii. (1 point) Explain clearly and precisely how you prove the invalidity of the argument using your counterexample.

2. (5 points) Prove using mathematical induction that for all integers $n \geq 1$ it holds that: $\sum_{i=1}^{n}(i \cdot i\,!) = (n+1)\,!-1$.

   Hint: Set up your proof such that it shows your *insight* in the steps of a proof by induction: The proof which you deliver should show indubitably that you understand how and why such a proof *works*! Give ample explanations and comments.

End of exam questions