Midterm Reasoning and Logic (TI1306)

- The use of the book, notes, calculators or other sources is **strictly prohibited**.

Multiple choice questions

1. Which formula shows that both p and q are necessary for r?
   A. $((p \wedge q) \rightarrow r)$.
   B. $(\neg(p \wedge q) \vee \neg r)$.
   C. $(\neg r \vee (p \wedge q))$**.**
   D. $(r \rightarrow \neg(p \wedge q))$.

Answer:

$$\begin{aligned}(\neg(p \wedge q) \rightarrow \neg r) &\equiv (r \rightarrow (p \wedge q)) \\ &\equiv (\neg r \vee (p \wedge q))\end{aligned}$$

2. Suppose you create a truth table for A and B, both formulas in propositional calculus, and have a look at the columns below the main connectives of A and B. When do we know **for sure** that $A \equiv B$ is true?
   A. When there is a T in every row under B if there is a T in the same row under A.
   B. If there is no row for which there is an F under A and a T under B.
   C. If there is a T in every row under A and a T in every row under B.
   D. If there is no row in which there is only a T under A or B.

Answer: A and B are equivalent if $(A \leftrightarrow B)$ is a tautology, so they are not equivalent if $(A \leftrightarrow B)$ is not a tautology, so if the columns under A and B are not identical.

3. XOR (the exclusive OR) $(p \oplus q)$ is equivalent with the **negation** of:
   A. the bi-implication $(p \leftrightarrow q)$**.**
   B. the conjunction $(p \wedge q)$.
   C. the NAND $(p \mid q)$.
   D. the implication $(p \rightarrow q)$.

Answer: This follows from the truth table: $p \leftrightarrow q$ is true precisely when $p \oplus q$ is false.

4. If p and q are propositions, what is the right way to prove that $(p \rightarrow q)$ is true?
   A. Assume p is true. (*Assume q is true.*) This means that $(p \rightarrow q)$ is true. QED
   B. Assume q is true. (*Now show that $\neg p$ is true*) This means that $(p \rightarrow q)$ is true. QED
   C. Assume q and $\neg p$ are true. (*Now deduce a contradiction*) This means that $p \rightarrow q$ is true. QED.
   D. Assume $\neg q$ is true. (Deduce that $\neg p$ is true.) This means $(p \rightarrow q)$ is true. QED.

Answer: One way of proving an implication is to prove its contrapositive. That is exactly what answer D means. Answer A is wrong because we cannot also assume that q is true, we have to deduce this instead.

5. Suppose $(\neg p \vee q)$ is true, $(q \rightarrow \neg r)$ is true, and r is true. What can we conclude?
   A. $(p \wedge \neg q)$ is true.
   B. $(p \rightarrow \neg q)$ is true.
   C. $(\neg p \wedge q)$ is true.
   D. $(\neg p \rightarrow q)$ is true.

Answer: Since r and $(q \rightarrow \neg r)$ are both true, we get $\neg q$ via the contrapositive. Combined with $(\neg p \vee q)$ this yields $\neg p$. Now answer B, which is equivalent to $(\neg p \vee \neg q)$, is the only possible answer.

6. Looking at the truth tables of formulas A, B, C in propositional calculus, when do you know for **sure** that the argument $A, B \therefore C$ is **not** logically valid?
   A. If there is a T under C in all rows where there is an F under A and B.
   B. If there is not a row for which there is a T under A, B and C.
   C. If there is an F under C in all rows where there is a T under A or B.
   D. If there is a row with a T under A and B, and an F under C.

Answer: $A, B \therefore C$ means that if A and B are both true, C should also be true. Thus, to show the formula is **not valid** we have to find a row in which the premises are true and the conclusion is false.

7. Simon and Charles are students in Delft. When can they know for sure that the next statement is **false**? "If there exists a student in Delft without a bicycle, then all students in Delft have a car."
   A. If Simon does not have a bicycle, but Charles has a car.
   B. If Charles does not have a bicycle, and also no car.
   C. If Charles has a bicycle, and Simon a car.
   D. If Simon has a bicycle, but not a car.

Answer: If $D(x)$ means "x is a student in Delft", $B(x)$ means "x has a bicycle", and $C(x)$ "x has a car", then we can translate the statement as follows:

$$(\exists x(D(x) \wedge \neg B(x)) \rightarrow \forall x(D(x) \rightarrow C(x))).$$

The negation of this is:

$$\begin{aligned}
&\neg(\exists x(D(x) \wedge \neg B(x)) \rightarrow \forall x(D(x) \rightarrow C(x))) \\
&\equiv \exists x(D(x) \wedge \neg B(x)) \wedge \neg \forall x(D(x) \rightarrow C(x)) \\
&\equiv \exists x(D(x) \wedge \neg B(x)) \wedge \exists x \neg(D(x) \rightarrow C(x)) \\
&\equiv \exists x(D(x) \wedge \neg B(x)) \wedge \exists x(D(x) \wedge \neg C(x))
\end{aligned}$$

Thus, the negation is true if and only if there exists a student in Delft without a bicycle and there exists a student in Delft without a car.

8. If $P(x, y)$ is a property of a pair of two integers, how can we prove the next statement?
"For all integers x there exists an integer y such that $P(x, y)$ is true.
   - A. Find an integer x such that $P(x, y)$ is true for an arbitrary integer y.
   - **B. Take an arbitrary integer x. Find an integer y such that $P(x, y)$ is true.**
   - C. Find an integer x, and an integer y such that $P(x, y)$ is true.
   - D. Take arbitrary x and y, and show that if $P(x, y)$ is true, then x and y cannot be integers.

Answer: The statement is

$$\forall x \in \mathbb{Z}\, \exists y \in \mathbb{Z}\, P(x, y).$$

To prove this, you need to show that for an arbitrary integer x, you can find an integer y such that $P(x, y)$ is true.

9. How can we show that the statement in question 8 is false?
   - A. Find an integer x and an integer y such that $P(x, y)$ is false.
   - B. Take an arbitrary integer x, and find an integer y such that $P(x, y)$ is false.
   - **C. Find an integer x, such that $P(x, y)$ is false for an arbitrary integer y.**
   - D. Take arbitrary integers x and y, and show that $P(x, y)$ is false.

Answer:

$$\begin{aligned} &\neg\forall x \in \mathbb{Z}\, \exists y \in \mathbb{Z}\, P(x, y) \\ &\equiv \exists x \in \mathbb{Z}\, \neg\exists y \in \mathbb{Z}\, P(x, y) \\ &\equiv \exists x \in \mathbb{Z}\, \forall y \in \mathbb{Z}\, \neg P(x, y), \end{aligned}$$

so we need to find an integer x such that for all integers y, $P(x, y)$ is false.

10. Which of the formulas below shows that there exists **exactly one** object with property P?
   - **A.** $\exists x(P(x) \wedge \forall y(P(y) \to (x = y)))$
   - B. $\exists x \forall y(P(x) \wedge (P(y) \to \neg(x = y)))$
   - C. $\exists x(P(x) \wedge \exists y(P(y) \to \neg(x = y)))$
   - D. $\exists x \exists y((P(x) \wedge P(y)) \to (x = y))$

Answer:
'There exists exactly one object with property P' means that there exists an object with property P ($\exists x P(x)$) and that there does not exist another object with property P, so a y with property P not equal to x does not exist: $\neg\exists y(P(y) \wedge \neg(x = y))$. A formula which expresses what we want is $\exists x(P(x) \wedge \neg\exists y(P(y) \wedge \neg(x = y)))$. We can rewrite this formula.

$$\begin{aligned} \exists x(P(x) \wedge \neg\exists y(P(y) \wedge \neg(x = y))) &\equiv \exists x(P(x) \wedge \forall y \neg(P(y) \wedge \neg(x = y))) \\ &\equiv \exists x(P(x) \wedge \forall y(\neg P(y) \vee \neg\neg(x = y))) \\ &\equiv \exists x(P(x) \wedge \forall y(\neg P(y) \vee (x = y))) \\ &\equiv \exists x(P(x) \wedge \forall y(P(y) \to (x = y))) \qquad \text{This is answer C} \end{aligned}$$

Open questions

1. Suppose we have the following argument.

   Argument. $((p \wedge q) \rightarrow \neg r), (p \vee \neg q), (\neg q \rightarrow p) \therefore \neg r.$

   (a) (2 points) Create a truth table to investigate if the argument is logically valid or not. (You can use either 0/1 or T/F. Order the rows in a systematic manner in any case.)

   > **Answer:** Should not be a problem, consists of 8 rows.

   (b) (1 point) Is the argument valid?

   > **Answer:** No.

   (c) (1 point) Explain clearly why your answer at question b follows from your truth table at question a. Give numbers to all relevant columns in the truth table and refer to these numbers in your explanation.

   > **Answer:** The participant should point out that there should be at least one row in the truth table in which the premises are true and the conclusion is false. For instance, in this case one could take p and r to be true and q to be false.

2. Suppose we have the following arguments.

   Argument (I). *For all x and y: if both x and y are* irrational, $x \cdot y$ *is also* irrational.

   Argument (II). *For all integers n: if n^3 is* even, *then n is* even.

   (a) Argument I is not valid.

   i. (1 point) Give a counterexample which shows that argument I is not valid. (Tip: in the lecture a famous irrational number came by.)

   > **Answer:** One or two irrational numbers should be mentioned here for which the product is rational, for example $x = y = \sqrt{2}$, then $x \cdot y = 2$.

   ii. (1 point) Explain clearly how your counterexample shows that the argument is not valid.

   > **Answer:** The participant should show here that the statement that says 'for all x and y' some other statement holds, is false when at least one x and y can be found for which the other statement is false. The implication is false if the antecedent is true and the consequent is false, so if two irrational numbers can be found such that $x \cdot y$ is not irrational. In this case, $x = y = \sqrt{2}$ satisfy this requirement.

   (b) (2 points) Argument II is valid. Give a proof. Clearly justify the steps in your proof. Alternate these steps with an explanation of what should be proven. (Tip: contrapositive.)

   > **Answer:** A proof should be given here for a 'for all' statement. The method of 'generalizing from the generic particular' can be used (1 point), so an arbitrary integer (for example k) should be used. If x or n are used, no points will be deducted. For this number k the implication should be proven (1 point), for example by proving the contrapositive. An example proof, using the contrapositive, is as follows:
   >
   > *Proof.* We need to prove that for all integers n, if n^3 is even, then n is even. To do this, we prove the contrapositive: for all integers n, if n is odd, then n^3 is odd. We need to prove a universally quantified statement, so take a random integer, say k. We now need to show that if k is odd, then k^3 is odd. We prove an implication by assuming the antecedent and proving the

consequent, so suppose k is indeed odd. This means we can write k as $k = 2m + 1$ for some integer m. If we now take the cube, we get

$$k^3 = (2m + 1)^3 = 8m^3 + 12m^2 + 6m + 1 = 2(4m^3 + 6m^2 + 3m) + 1 = 2\ell + 1,$$

for an integer $\ell = 4m^3 + 6m^2 + 3m$. This means k^3 is also odd, which we had to prove. Since k is a random integer, it now holds for all integers n that if n^3 is even, then n is even. QED

End of the exam