Toets 1 Wiskundige Structuren TW1010
Test 1 Mathematical Structures TW1010
Woensdag 4 oktober 2017 / Wednesday October 4, 2017
11.00-12.00

Naam / Name: [ ]

Studienummer / Student number: [| | | | | |]

Rekenmachines zijn niet toegestaan. Vul de antwoorden in de vakken in. Het cijfer is (score+4)/4.
No calculators allowed. Write the solutions in the fields provided. The grade is (score+4)/4.

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1 Laat met behulp van een waarheidstabel zien dat de volgende uitspraak een tautologie is: 3
Show using a truth table that the following statement is a tautology:

$$(p \Rightarrow q) \vee (q \Rightarrow p)$$

p	q	$(p \Rightarrow q)$	$\vee$	$(q \Rightarrow p)$
T	T	T	T	T
T	F	F	T	T
F	T	T	T	F
F	F	T	T	T

Beneath the $\vee$ there are only T's for true,
thus the statement is a tautology.

2a Vereenvoudig de negatie van / Simplify the negation of 4

$$\forall x \in \mathbb{Z} : \exists y \in \mathbb{R} : x > y \Rightarrow (x > 0 \vee x = y + 1)$$

zoveel als mogelijk. U mag geen negatie-tekens laten staan. Bij uitzondering is uitleg hierbij niet nodig. / As much as possible. No negation symbols should remain. By exception no explanation is necessary for this exercise.

$$\exists x \in \mathbb{Z} : \forall y \in \mathbb{R} : (x > y) \wedge (x \leq 0) \wedge (x \neq y + 1)$$

2b Bewijs dat de uitspraak van opgave 2a waar is. / Prove that statement from exercise 2a is true. 3

Let $x \in \mathbb{Z}$ be arbitrary. Take $y = x$.

Then the antecedent $x > y$ is false, so the implication is always true.

For all x we can find a y for which the resulting expression is true,

so the statement from 2a is true.

NB: An alternative proof can be found by taking $y = x - 1$.

3 Een relatie R op $\mathbb{N}$ is gedefinieerd door xRy als $xy = 3k$ voor een $k \in \mathbb{N}$.
A relation R on $\mathbb{N}$ is defined by xRy whenever $xy = 3k$ for some $k \in \mathbb{N}$.

3a Controleer of deze relatie symmetrisch is. / Determine whether this relation is symmetric. 4

Symmetric means $\forall x, y : xRy \Rightarrow yRx$.

Let $x, y \in \mathbb{N}$ be arbitrary and suppose xRy holds. Thus there is a $k \in \mathbb{N}$ with $xy = 3k$.

But then $yx = xy = 3k$ for the same $k \in \mathbb{N}$ and thus yRx holds as well.

Therefore this relation is symmetric.

3b Controleer of deze relatie transitief is. / Determine whether this relation is transitive. 4

Transitive means $\forall x, y, z : xRy \wedge yRz \Rightarrow xRz$.

Choose $x = 1$, $y = 3$ and $z = 1$. Then $xy = 3 = 3 \cdot 1$ is of the form $3k$ so xRy holds.

Similarly $yz = 3 = 3 \cdot 1$ is also of the form $3k$ and thus yRz also holds.

However, $xz = 1 = 3 \cdot \frac{1}{3}$ is not of the form $3k$ with $k \in \mathbb{N}$ and therefore xRz does not hold.

The relation is therefore not transitive.

4 A, B en C zijn willekeurige verzamelingen. / A, B, and C are arbitrary sets.

4a Laat zien dat/ Show that 5

$$C \setminus (A \cup B) \subseteq (C \setminus A) \cup (C \setminus B).$$

Let $x \in C \setminus (A \cup B)$ be arbitrary. Then $x \in C$ and $x \notin A \cup B$.

The latter implies that $x \notin A$ and $x \notin B$.

Since $x \in C$ and $x \notin A$ we find $x \in C \setminus A$.

Thus $x \in C \setminus A$ or $x \in C \setminus B$. Hence $x \in (C \setminus A) \cup (C \setminus B)$.

We have now proven the inclusion.

NB: You can even prove that $x \in (C \setminus A)$ en $x \in (C \setminus B)$.

So we have $C \setminus (A \cup B) \subseteq (C \setminus A) \cap (C \setminus B)$.

Indeed these two sets are equal.

4b Laat zien dat niet in het algemeen geldt / Show that it is not true in general that 3

$$C \setminus (A \cup B) = (C \setminus A) \cup (C \setminus B).$$

Take $A = C = \{1\}$ and $B = \emptyset$.

Then $A \cup B = \{1\}$ and $C \setminus (A \cup B) = \emptyset$.

The right hand side gives $C \setminus A = \emptyset$ and $C \setminus B = \{1\}$. Therefore $(C \setminus A) \cup (C \setminus B) = \{1\}$.

Both sides of the equation are unequal in this example, and thus equality does not hold in general.

5 Maak de volgende definitie af: Een functie $g : A \to B$ heet surjectief als Complete the following definition: A function $g : A \to B$ is called surjective if 2

for all $b \in B$ there exists $a \in A$ satisfying $g(a) = b$.

6 Formuleer het welordeningsaxioma voor $\mathbb{N}$. / Formulate the well-ordering property of $\mathbb{N}$. 2

Every non-empty subset of $\mathbb{N}$ has a smallest element.

7 Laat zien dat voor alle $n \in \mathbb{N}$ geldt dat / Show that for all $n \in \mathbb{N}$ we have 6

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} \geq \ln(n+1)$$

U mag gebruiken dat voor alle $x > -1$ geldt dat $x \geq \ln(1+x)$ en dat voor alle $a, b > 0$ geldt dat $\ln(a) + \ln(b) = \ln(ab)$. U mag geen integralen of afgeleiden gebruiken.
U may use that for $x > -1$ the inequality $x \geq \ln(1+x)$ holds, and that for all $a, b > 0$ the equation $\ln(a)+\ln(b) = \ln(ab)$ is valid. You are not allowed to use integrals or derivatives.

We prove the inequality using induction to n.

For $n = 1$ the left hand side is 1 and the right hand side $\ln(1+1) = \ln(2)$.

Since $1 \geq \ln(2)$, the inequality holds for $n = 1$.

Suppose the inequality holds for $n = k$, thus suppose $1 + \frac{1}{2} + \cdots + \frac{1}{k} \geq \ln(k+1)$.

Then we also have $1 + \frac{1}{2} + \cdots + \frac{1}{k} + \frac{1}{k+1} \geq \ln(k+1) + \frac{1}{k+1}$.

Hence $1 + \frac{1}{2} + \cdots + \frac{1}{k} + \frac{1}{k+1} \geq \ln(k+1) + \frac{1}{k+1} \geq \ln(k+1) + \ln(1 + \frac{1}{k+1})$.

$= \ln((k+1)(1 + \frac{1}{k+1})) = \ln(k+2)$.

Thus the inequality is also valid for $n = k+1$.

Using induction we have now shown that the inequality holds for all $n \in \mathbb{N}$.

Opgave / Exercise ☐ voortgezet (extra ruimte) / continued (extra space)

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